Technische Universität Berlin
Institut für Mathematik

Differential Geometry II

Analysis and geometry on manifolds

Prof. Dr. Ulrich Pinkall

Lecture notes
Dr. Felix Knöppel

November 18, 2020

Contents

1. n-Dimensional Manifolds

Informally, an n-dimensional manifold is a "space" which locally (when looked at through a microscope) looks like "flat space" $\mathbb{R}^n$.
Many important examples of manifolds M arise as certain subsets $M \subset \mathbb{R}^k$, e.g.:

1. n-dimensional affine subspaces $M \subset \mathbb{R}^k$
2. $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}$
3. compact 2-dimensional submanifolds of $\mathbb{R}^3$
4. $SO(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^t A = Id\}$ is a 3-dimensional submanifold of $\mathbb{R}^9$

Flat spaces (vector spaces $\cong \mathbb{R}^n$) are everywhere. Curved manifolds come up in Stochastics, Algebraic Geometry, ..., Economics and Physics – e.g. as the configuration space of a pendulum ($\mathbb{S}^2$), a double pendulum ($\mathbb{S}^2 \times \mathbb{S}^2$) or rigid body motion (SO(3)), or as space time in general relativity (the curved version of flat special relativity).

1.1 Crash Course in Topology

Definition 1.1 (Topological space)**.** *A topological space is a set* M *together with a subset* $\mathcal{O} \subset \mathcal{P}(M)$ *(the collection of all "open sets") such that:*

1. $\emptyset, M \in \mathcal{O}$,
2. $U_\alpha \in \mathcal{O}$, $\alpha \in I \Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \mathcal{O}$,
3. $U_1, \ldots, U_n \in \mathcal{O}, \Rightarrow U_1 \cap \cdots \cap U_n \in \mathcal{O}$.

Remark 1.2. Usually we suppress the the collection $\mathcal{O}$ of open sets and just say M is a topological space. If several topologies and spaces are involved we use an index to make clear which topology corresponds to which space.

Some ways to make new topological spaces out of given ones:

a) Let X be a topological space, $M \subset X$, then $\mathcal{O}_M := \{U \cap M \mid U \in \mathcal{O}_X\}$ defines a topology on M – called "induced topology" or "subspace topology".

b) Let X be a topological space, M be a set, and $\pi\colon X \to M$ a surjective map. Then there is a bijection between M and the set of equivalence classes of the equivalence relation on X defined by
$$x \sim y \Leftrightarrow \pi(x) = \pi(y).$$

In other words: M can be identified with the set of equivalence classes.
Conversely, given an equivalence relation $\sim$ on a topological space X we can form the set of equivalence classes $M = X/_{\sim}$. The *canonical projection* $\pi\colon X \to M$ is the surjective map which sends $x \in X$ to the corresponding equivalence class $[x]$. The *quotient topology*

$$\mathcal{O}_M = \left\{ U \subset M \mid \pi^{-1}(U) \in \mathcal{O}_X \right\}$$

turns M into a topological space. By construction π is continuous.

Exercise 1.3 (Product topology)**.**
Let M and N be topological spaces and define

$$\mathcal{B} := \{U \times V \mid U \in \mathcal{O}_M, V \in \mathcal{O}_N\}$$

Show that

$$\mathcal{O} := \left\{ \bigcup_{U \in \mathcal{A}} U \mid \mathcal{A} \subset \mathcal{B} \right\}$$

is a topology on $M \times N$.

Definition 1.4 (Continuity)**.** *Let* M, N *be topological spaces. Then* $f\colon M \to N$ *is called continuous if*

$$f^{-1}(U) \in \mathcal{O}_M \text{ for all } U \in \mathcal{O}_N.$$

Definition 1.5 (Homeomorphism)**.** *A bijective map* $f\colon M \to N$ *between topological spaces is called a homeomorphism if* f *and* f^{-1} *are both continuous.*

Remark 1.6. If $f\colon M \to N$ is a homeomorphism, then for

$$U \in \mathcal{O}_M \Leftrightarrow f(U) \in \mathcal{O}_N$$

So two topological spaces are topologically indistinguishable, if they are homeomorphic, i.e. if there exists a homeomorphism $f\colon M \to N$.

Definition 1.7 (Hausdorff)**.** *A topological space* M *is called Hausdorff if for all* $x, y \in M$ *with* $x \neq y$ *there are open sets* $U_x, U_y \in \mathcal{O}$ *with* $U_x \cap U_y = \emptyset$.

Example 1.8. The quotient space $M = \mathbb{R}/\sim$ with $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ is not Hausdorff.

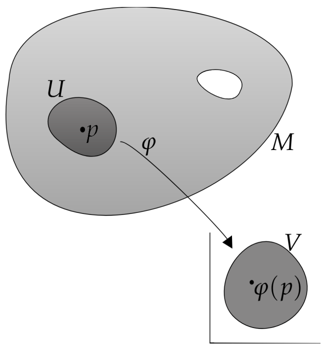
Definition 1.9 (Second axiom of countability)**.** *A topological space* M *is said to satisfy the second axiom of countability (or is called second countable), if there is a countable base of topology, i.e. there is a sequence of open sets* $U_1, U_2, U_3, \ldots \in \mathcal{O}$ *such that for every* $U \in \mathcal{O}$ *there is a subset* $I \subset \mathbb{N}$ *such that* $U = \bigcup_{\alpha \in I} U_\alpha$.

Example 1.10. The balls of rational radius with rational center in $\mathbb{R}^n$ form a countable base of topology, i.e. $\mathbb{R}^n$ is 2nd countable.

Remark 1.11. Subspaces of second countable spaces are second countable. Hence all subsets of $\mathbb{R}^n$ are second countable. A similar statement holds for the Hausdorff property.

Example 1.12. $\mathrm{M} = \mathbb{R}^2$ with the topology generated by $\mathcal{B} = \{U \times \{y\} \mid y \in \mathbb{R}, U \in \mathcal{O}_{\mathbb{R}}\}$ is not second countable.

Definition 1.13 (Topological manifold)**.** *A topological space* M *is called an n-dimensional topological manifold if it is Hausdorff, second countable and for every* $p \in \mathrm{M}$ *there is an open set* $U \in \mathcal{O}$ *with* $p \in U$ *and a homeomorphism* $\varphi\colon U \to V$*, where* $V \in \mathcal{O}_{\mathbb{R}^n}$*.*



Definition 1.14 (coordinate chart)**.** *Let* M *be an n-dimensional topological manifold. Then a coordinate chart of* M *is a pair* (U, φ)*, where* $U \subset M$ *is open and* $\varphi\colon U \to V \subset \mathbb{R}^n$ *is a homeomorphism onto an open set* $V \subset \mathbb{R}^n$*.*

Exercise 1.15.
Let X be a topological space, $x \in \mathrm{X}$ and $n \geq 0$. Show that the following statements are equivalent:

i) There is a neighborhood of x which is homeomorphic to $\mathbb{R}^n$.

ii) There is a neighborhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$.

Exercise 1.16.
Show that a manifold M is locally compact, i.e. each point of M has a compact neighborhood.

Definition 1.17 (Connectedness)**.** *A topological space* X *is connected if the only subsets of* X *which are simultaneously open and closed are* X *and* $\emptyset$*.*
Moreover, X *is called path-connected if any two points* $x, y \in \mathrm{X}$ *can be joined by a path, i.e. there is a continuous map* $\gamma\colon [0,1] \to \mathrm{X}$ *such that* $\gamma(0) = x$ *and* $\gamma(1) = y$*.*

Exercise 1.18.
Show that a manifold is connected if and only it is path-connected.

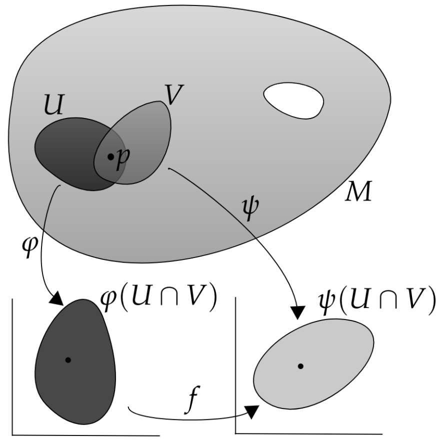
Definition 1.19 (coordinate change)**.** *Given two charts* $\varphi\colon U \to \mathbb{R}^n$ *and* $\psi\colon V \to \mathbb{R}^n$*, then the map*

$$f\colon \varphi(U\cap V) \to \psi(U\cap V)$$

given by

$$f = \psi \circ (\varphi|_{U\cap V})^{-1}$$

is a homeomorphism, called the coordinate change or transition map.



Definition 1.20 (Atlas)**.** *An atlas of a manifold* M *is a collection of charts* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ *such that* $\mathrm{M} = \bigcup\limits_{\alpha\in I} U_\alpha$*.*

1.2 Smooth Manifolds

Definition 1.21 (Compatible charts)**.** *Two charts*

$$\varphi\colon U \to \mathbb{R}^n, \quad \psi\colon V \to \mathbb{R}^n$$

on a topological manifold M *are called compatible if*

$$f\colon \varphi(U\cap V) \to \psi(U\cap V)$$

is a diffeomorphism, i.e. f *and* f^{-1} *both are smooth.*

Example 1.22. Consider $\mathrm{M} = \mathbb{S}^n \subset \mathbb{R}^{n+1}$, let $B = \{y \in \mathbb{R}^n \mid \|y\| \leq 1\}$ an define charts as follows:
For $i = 0, \dots, n$,

$$U_i^{\pm} = \{x \in \mathbb{S}^2 \mid \pm x_i > 0\}, \quad \varphi_i^{\pm}, \colon U_i^{\pm} \to B, \quad \varphi_i^{\pm}(x_0, \dots, x_n) = (x_0, \dots, \widehat{x_i}, \dots, x_n),$$

where the hat means omission. To check that φ_i are homeomorphisms is left as an exercise. So: (Since $\mathbb{S}^n$ as a subset of $\mathbb{R}^{n+1}$ is Hausdorff and second countable) $\mathbb{S}^n$ is an n-dimensional topological manifold. All $\varphi_i^{\pm}$ are compatible, so this atlas turns $\mathbb{S}^n$ into a smooth manifold.

An atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ of mutually compatible charts on M is called *maximal* if every chart (U, φ) on M which is compatible with all charts in $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ is already contained in the atlas.

Definition 1.23 (Smooth manifold)**.** *A differentiable structure on a topological manifold* M *is a maximal atlas of compatible charts. A smooth manifold is a topological manifold together with a maximal atlas.*

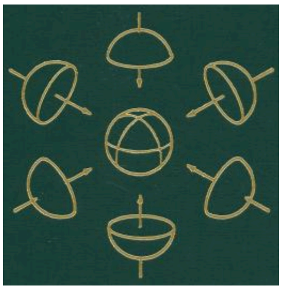


Figure 1.1: This illustration for the case $n = 2$ is taken from the title page of the book "Riemannian Geometry" by Manfredo do Carmo (Birkenhäuser 1979).

Exercise 1.24 (Real projective space)**.**
Let $n \in \mathbb{N}$ and $X := \mathbb{R}^{n+1} \setminus \{0\}$. The quotient space $\mathbb{RP}^n = X/_\sim$ with equivalence relation given by

$$x \sim y :\Longleftrightarrow x = \lambda y, \quad \lambda \in \mathbb{R}$$

is called the n-dimensional *real projective space*. Let $\pi\colon X \to \mathbb{RP}^n$ denote the *canonical projection* $x \mapsto [x]$.
For $i = 0, \ldots, n$, we define $U_i := \pi(\{x \in X \mid x_i \neq 0\})$ and $\varphi_i\colon U_i \to \mathbb{R}^n$ by

$$[x_0, \ldots, x_n] \mapsto (x_0/x_i, \ldots, \widehat{x_i}, \ldots, x_n/x_i).$$

Show that

a) π is an *open map*, i.e. maps open sets in X to open sets in $\mathbb{RP}^n$,

b) the maps φ_i are well-defined and $\{(U_i, \varphi_i)\}_{i\in I}$ is a smooth atlas of $\mathbb{RP}^n$,

c) $\mathbb{RP}^n$ is compact. **Hint:** Note that the restriction of π to $\mathbb{S}^n$ is surjective.

Exercise 1.25 (Product manifolds)**.**
Let M and N be topological manifolds of dimension m and n, respectively. Show that their Cartesian product $M \times N$ is a topological manifold of dimension $m + n$.
Show further that, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is a smooth atlas of M and $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$ is a smooth atlas of N, then $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{(\alpha,\beta)\in A\times B}$ is a smooth atlas of $M \times N$. Here $\varphi_\alpha \times \psi_\beta\colon U_\alpha \times V_\beta \to \varphi_\alpha(U_\alpha) \times \psi_\beta(V_\beta)$ is defined by $\varphi_\alpha \times \psi_\beta(p, q) := (\varphi_\alpha(p), \psi_\beta(q))$.

Exercise 1.26 (Torus)**.**
Let $\mathbb{R}^n/\mathbb{Z}^n$ denote the quotient space $\mathbb{R}^n/_\sim$ where the equivalence relation is given by

$$x \sim y :\Leftrightarrow x - y \in \mathbb{Z}^n.$$

Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $x \mapsto [x]$ denote the canonical projection. Show:

a) π is a *covering map*, i.e. a continuous surjective map such that each point $p \in \mathbb{R}^n/\mathbb{Z}^n$ has a open neighborhood V such that $\pi^{-1}(V)$ is a disjoint union of open sets each of which is mapped by π homeomorphically to V.

b) π is an open map.

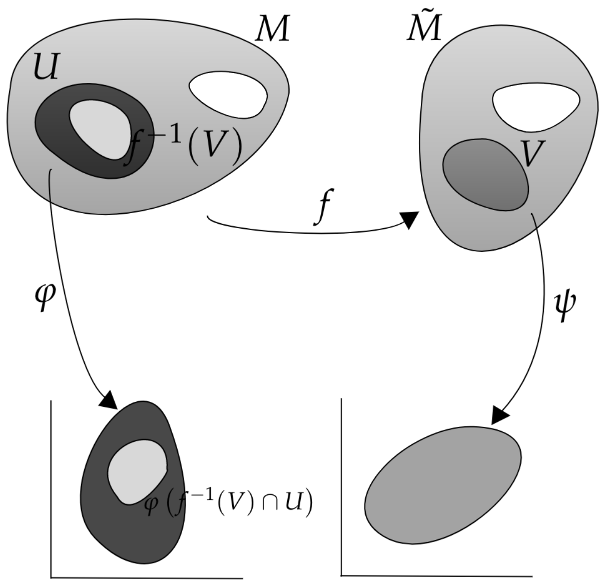
c) $\mathbb{R}^n/\mathbb{Z}^n$ is a manifold of dimension n.

d) $\{(\pi|_U)^{-1} \mid U \subset \mathbb{R}^n \textit{ open},\ \pi|_U : U \to \pi(U) \textit{ bijective}\}$ is a smooth atlas of $\mathbb{R}^n/\mathbb{Z}^n$.

Definition 1.27 (Smooth map)**.** *Let* M *and* $\tilde{\mathrm{M}}$ *be smooth manifolds. Then a continuous map* $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ *is called smooth if for every chart* (U,φ) *of* M *and every chart* (V,ψ) *of* $\tilde{\mathrm{M}}$ *the map*

$$\varphi(f^{-1}(V) \cap U) \to \psi(V), \quad x \mapsto \psi(f(\varphi^{-1}(x)))$$

is smooth.



Definition 1.28 (Diffeomorphism)**.** *Let* M *and* $\tilde{\mathrm{M}}$ *be smooth manifolds. Then a bijective map* $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ *is called a diffeomorphism if both* f *and* f^{-1} *are smooth.*

One important task in Differential Topology is to classify all smooth manifolds up to diffeomorphism.

Example 1.29. Every connected one-dimensional smooth manifold is diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. Examples of 2-dimensional manifolds: [pictures missing: compact genus 0,1,2,... Klein bottle, or torus with holes (non-compact)] - gets much more complicated already. For 3-dimensional manifolds there is no list.

Exercise 1.30. Show that the following manifolds are diffeomorphic.

a) $\mathbb{R}^2/\mathbb{Z}^2$.

b) the product manifold $\mathbb{S}^1 \times \mathbb{S}^1$.

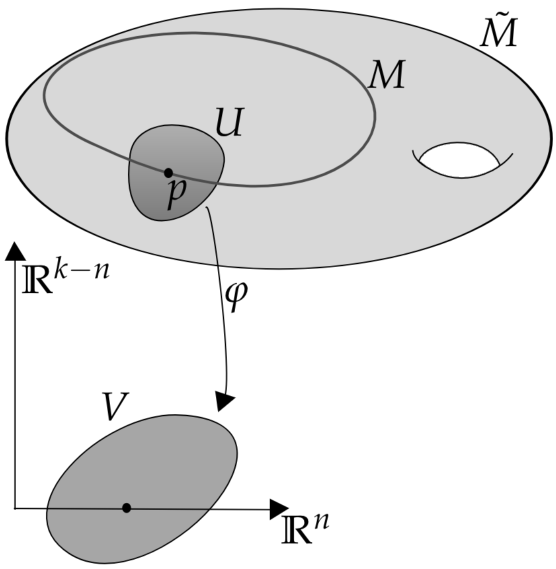
c) the torus of revolution as a submanifold of $\mathbb{R}^3$:

$$T = \big\{((R + r\cos\varphi)\cos\theta, (R + r\cos\varphi)\sin\theta, r\sin\varphi) \mid \varphi, \theta \in \mathbb{R}\big\}.$$

1.3 Submanifolds

Definition 1.31 (Submanifold)**.** *A subset* $\mathrm{M} \subset \tilde{\mathrm{M}}$ *in a k-dimensional smooth manifold* $\tilde{\mathrm{M}}$ *is called an n-dimensional submanifold if for every point* $p \in \mathrm{M}$ *there is a chart* $\varphi\colon U \to V$ *of* $\tilde{\mathrm{M}}$ *with* $p \in U$ *such that*

$$\varphi(U \cap \mathrm{M}) = V \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^k.$$



Let us briefly restrict attention to $\tilde{\mathrm{M}} = \mathbb{R}^k$.

Theorem 1.32. *Let* $\mathrm{M} \subset \mathbb{R}^k$ *be a subset. Then the following are equivalent:*

a) M *is an n-dimensional submanifold,*

b) locally M *looks like the graph of a map from* $\mathbb{R}^n$ *to* $\mathbb{R}^{k-n}$*, which means: For every point* $p \in \mathrm{M}$ *there are open sets* $V \subset \mathbb{R}^n$ *and* $W \subset \mathrm{M}$, $W \ni p$*, a smooth map* $f\colon V \to \mathbb{R}^{k-n}$ *and a coordinate permutation* $\pi\colon \mathbb{R}^k \to \mathbb{R}^k$, $\pi(x_1, ..., x_k) = (x_{\sigma_1}, ..., x_{\sigma_k})$ *such that*

$$\pi(W) = \{(x, f(x)) \mid x \in V\},$$

c) locally M *is the zero set of some smooth map into* $\mathbb{R}^{k-n}$*, which means: For every* $p \in \mathrm{M}$ *there is an open set* $U \subset \mathbb{R}^k$, $U \ni p$ *and a smooth map* $g\colon U \to \mathbb{R}^{k-n}$ *such that*

$$\mathrm{M} \cap U = \{x \in U \mid g(x) = 0\}$$

and the Jacobian $g'(x)$ *has full rank for all* $x \in \mathrm{M} \cap U$*,*

d) locally M *can be parametrized by open sets in* $\mathbb{R}^n$*, which means: For every* $p \in \mathrm{M}$ *there are open sets* $W \subset \mathrm{M}$, $W \ni p$, $V \subset \mathbb{R}^n$ *and a smooth map* $\psi\colon V \to \mathbb{R}^k$ *such that* ψ *maps* V *bijectively onto* W *and* $\psi'(x)$ *has full rank for all* $x \in V$*.*

Remark 1.33.
First, recall two theorems from analysis:

The inverse function theorem:
Let $U \subset \mathbb{R}^n$ be open, $p \in U$, $f\colon U \to \mathbb{R}^n$ continuously differentiable, $\det f'(p) \neq 0$. Then there is a an open' subset $\tilde{U} \subset U$, $\tilde{U} \ni p$ and an open subset $V \subset \mathbb{R}^n$, $V \ni f(p)$ such that

1. $f|_{\tilde{U}} : \tilde{U} \to V$ is bijective,
2. $f|_{\tilde{U}}^{-1} : V \to \tilde{U}$ is continuously differentiable.

We have $(f^{-1})'(q) = f'(f^{-1}(q))^{-1}$ for all $q \in V$. We in fact need a version where 'continuously differentiable' is replaced by $\mathscr{C}^\infty$. Let us prove the $\mathscr{C}^2$ version. Then all the partial derivatives of first order for f^{-1} are entries of $(f^{-1})'$. So we have to prove that $q \mapsto (f^{-1})'(q) = (f')^{-1}(f^{-1}(q))$ is continuously differentiable. This follows from the smoothness of the map $\mathrm{GL}(n, \mathbb{R}) \ni A \mapsto A^{-1} \in \mathrm{GL}(n, \mathbb{R})$ (Cramer's rule), the chain rule and the fact that $f'\colon \tilde{U} \to \mathbb{R}^{n \times n}$ is continuously differentiable. The general case can be done by induction.

The implicit function theorem ($\mathscr{C}^\infty$ *− version***)**
Let $U \subset \mathbb{R}^k$ be open, $p \in U$, $g\colon U \to \mathbb{R}^{k-n}$ smooth, $g(p) = 0$, $g'(p)$ is surjective. Then, after reordering the coordinates of $\mathbb{R}^k$, we find open subsets $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^{n-k}$ such that $(p_1, \ldots, p_n) \in V$ and $(p_{n+1}, \ldots, p_k) \in W$ and $V \times W \subset U$. Moreover, there is a smooth map $f\colon V \to W$ such that $\{q \in V \times W \mid g(q) = 0\} = \{(x, f(x)) \mid x \in V\}$.

Proof. (of Theorem 1.32)

(b) $\Rightarrow$ **(a):** Let $p \in \mathrm{M}$. By *b*) after reordering coordinates in $\mathbb{R}^k$ we find open sets $V \in \mathbb{R}^n$, $W \subset \mathbb{R}^{k-n}$ such that $p \in V \times W$ and we find a smooth map $f\colon V \to W$ such that

$(V \times W) \cap \mathrm{M} = \{(x, f(x)) \mid x \in V\}$. Then

$$\varphi\colon V \times W \to \mathbb{R}^k, (x,y) \mapsto (x, y - f(x))$$

is a diffeomorphism and $\varphi(\mathrm{M} \cap (V \times W)) \subset \mathbb{R}^n \times \{0\}$.

(a) $\Rightarrow$ (c): Let $p \in \mathrm{M}$. By $a)$ we find an open $U \in \mathbb{R}^k$, $U \ni p$ and a diffeomorphism $\varphi\colon U \to \hat{U} \subset \mathbb{R}^k$ such that $\varphi(U \cap \mathrm{M}) \subset \mathbb{R}^n \times \{0\}$. Now define $g\colon U \to \mathbb{R}^{k-n}$ to be the last $k-n$ component functions of φ, i.e. $\varphi = (\varphi_1, \dots, \varphi_n, g_1, \dots, g_{k-n})$. Then $\mathrm{M} \cap (V \times W) = g^{-1}(\{0\})$. For $q \in V \times W$ we have

$$\varphi'(q) = \begin{pmatrix} * \\ \vdots \\ * \\ g_1'(q) \\ \vdots \\ g_{k-n}'(q) \end{pmatrix}.$$

Hence g' has rank $k-n$.

(c)$\Rightarrow$ (b): This is just the implicit function theorem.

(b)$\Rightarrow$ (d): Let $p \in \mathrm{M}$. After reordering the coordinates by $b)$ we have an open neighborhood of p of the form $V \times W$ and a smooth map $f\colon V \to W$ such that

$$\mathrm{M} \cap (V \times W) = \{(x, f(x)) \mid x \in V\}.$$

Now define $\psi\colon V \to \mathbb{R}^k$ by $\psi(x) = (x, f(x))$, then ψ is smooth

$$\psi'(x) = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^n} \\ f'(x) \end{pmatrix}$$

So $\psi'(x)$ has rank n for all $x \in V$. Moreover, $\psi(V) = \mathrm{M} \cap (V \times W)$.

(d) $\Rightarrow$ (b): Let $p \in \mathrm{M}$. Then by $d)$ there are open sets $\hat{V} \subset \mathbb{R}^n$, $U \subset \mathbb{R}^k$, $U \ni p$ and a smooth map $\psi\colon \hat{V} \to \mathbb{R}^k$ such that $\psi(\hat{V}) = \mathrm{M} \cap U$ such that rank $\psi'(x)$ is n for all $x \in \hat{V}$. After reordering the coordinates on $\mathbb{R}^k$ we can assume that $\psi = (\phi, \hat{f})^t$ with $\phi\colon \hat{V} \to \mathbb{R}^n$ with $\det \phi'(x_0) \neq 0$, where $\psi(x_0) = p$. Passing to a smaller neighborhood $V \subset \hat{V}$, $V \ni p$, we then achieve that $\phi\colon V \to \phi(V)$ is a diffeomorphism (by the inverse function theorem). Now for all $y \in \phi(V)$ we have

$$\psi(\phi^{-1}(y)) = \begin{pmatrix} \phi(\phi^{-1}(y)) \\ \hat{f}(\phi^{-1}(y)) \end{pmatrix} =: \begin{pmatrix} y \\ f(\phi^{-1}(y)) \end{pmatrix}$$

□

1.3.1 Examples of submanifolds in $\mathbb{R}^k$

n-dimensional unit sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

is an n-dimensional submanifold (a *hypersurface*) of $\mathbb{R}^{n+1}$, because

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid g(x) = 0\}$$

where

$$g \colon \mathbb{R}^{n+1} \to \mathbb{R},\ g(x) = x_1^2 + x_2^2 + \cdots x_{n+1}^2 - 1$$

We now have to check that $g'(x)$ has rank 1 on $g^{-1}(\{0\})$: We easily see that $g'(x) = 2x \neq 0$ for $x \neq 0$.

orthogonal-group

$O(n) \subset \mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$ defined by

$$O(n) = \{A \in \mathbb{R}^{n \times n} \mid A^t A = I\}$$

is a submanifold of $\mathbb{R}^{n^2}$ of dimension $n(n-1)/2$.
Define

$$g \colon \mathbb{R}^{n \times n} \to \mathrm{Sym}(n) = \mathbb{R}^{n(n-1)/2},\ A \mapsto g(A) = A^t A - I$$

Then

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ * & a_{22} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ * & * & \cdots & a_{nn} \end{pmatrix}$$

The number of entries above and including the diagonal is

$$n + (n+1) + \cdots + 2 + 1 = n(n-1)/2$$

So we still need to check that $g'(A) \colon \mathbb{R}^{n \times n} \to \mathrm{Sym}(n)$ is surjective for all $A \in O(n)$.

Interlude: Consider derivatives of maps $f \colon U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^k$ open. Then $f'(p) \colon \mathbb{R}^k \to \mathbb{R}^m$ is linear. But how to calculate $f'(p)X$ for $X \in \mathbb{R}^k$?
Choose some smooth

$$\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^k$$

such that $\gamma(0) = p$ and $\gamma'(0) = X$. Then by the chain rule

$$(f \circ \gamma)'(0) = f'(\gamma(0))\gamma'(0) = f'(p)X.$$

So let $A \in O(n)$, $X \in \mathbb{R}^{n \times n}$, $B : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n \times n}$ with $B(0) = A$, $B'(0) = X$ (e.g. $B(t) = A + tX$).
Then

$$\begin{aligned} g'(A)X &= \frac{d}{dt}\Big|_{t=0} g(B(t)) \\ &= \frac{d}{dt}\Big|_{t=0} [B(t)^t B(t) - I] \\ &= (B^t)'(0)B(0) + B^t(0)B'(0) \\ &= X^t A + A^t X. \end{aligned}$$

which is the end of the interlude.
To check that $g'(A)$ is surjective, let $Y \in \mathrm{Sym}(n)$ be arbitrary. So $Y \in \mathbb{R}^{n\times n}$, $Y^t = Y$. There is $X \in \mathbb{R}^{n\times n}$ with $X^tA + A^tX = Y$, e.g. $X = \frac{1}{2}AY$:
By a straightforward calculation we yield

$$X^tA + A^tX = \tfrac{1}{2}(Y^tA^tA + A^tAY) = Y.$$

So $\mathrm{O}(n)$ is a submanifold dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

The Grassmanian of k-planes

Consider the set

$$G_k(\mathbb{R}^n) := \{k\text{-dimensional linear subspace}\}$$

the set of k-dimensional linear subspaces of $\mathbb{R}^n$. We represent a linear subspace $U \subset \mathbb{R}^k$ by the orthogonal projection $P_U \in \mathbb{R}^{n\times n}$ onto U. The map P_U is defined by

$$P_U|_U = \mathrm{Id}_U, \quad P_U|_{U^\perp} = 0 \tag{1.1}$$

P_U has the following properties:

$$P_U^2 = P_U, \quad P_U^* = P_U, \quad \operatorname{tr} P_U = \dim U$$

In the decomposition $\mathbb{R}^n = U \oplus U^\perp$, we have

$$P_U = \begin{pmatrix} Id_U & 0 \\ 0 & 0 \end{pmatrix}.$$

Conversely: If $P^* = P$, then there is an orthonormal basis of $\mathbb{R}^n$ with respect to which P is diagonal.

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

If further $P^2 = P$, then $\lambda_i^2 = \lambda_i \Leftrightarrow \lambda_i \in \{0,1\}$ for all $i \in \{1,\dots,n\}$. After reordering the basis we have

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

for some $k < n$. So P is the orthogonal projection onto a k-dimensional subspace with $k = \operatorname{tr} P$.
Thus we have

$$\mathrm{G}_k(\mathbb{R}^n) = \{P \in \mathrm{End}(\mathbb{R}^n) \mid P^2 = P,\, P^* = P,\, \operatorname{trace} P = k\}.$$

We fix a k-dimensional subspace V and define

$$W_V := \{L \in \mathrm{End}(\mathbb{R}^n) \mid P_V \circ L|_V \; invertible\}.$$

Since W_V is open, the intersection $\mathrm{G}_k(\mathbb{R}^n) \cap W_V$ is open in the subspace topology.

Fix a k-dimensional subspace $V \subset \mathbb{R}^n$. Then a k-dimensional subspace $U \subset \mathbb{R}^n$ 'close' to V is the graph of a linear map $Y \in \mathrm{Hom}(V, V^\perp)$:
With respect to the splitting $\mathbb{R}^n = V \oplus V^\perp$,

$$U = \mathrm{Im}\begin{pmatrix} \mathrm{Id}_V \\ Y \end{pmatrix} = \{(x, Yx) \mid x \in V\}.$$

The orthogonal complement $U^\perp$ of U is then parametrized over $V^\perp$ by $(-Y^*, \mathrm{Id}_{V^\perp})$. For $x \in V$ and $y \in V^\perp$ we have

$$\left\langle \begin{pmatrix} x \\ Yx \end{pmatrix}, \begin{pmatrix} -Y^*y \\ y \end{pmatrix} \right\rangle = \langle x, -Y^*y\rangle + \langle x, Yy\rangle = 0$$

Since rank $(-Y, \mathrm{Id}_{V^\perp})$ is $n-k$, we get

$$U^\perp = \mathrm{Im} \begin{pmatrix} -Y^* \\ \mathrm{Id}_{V^\perp} \end{pmatrix}.$$

Further, since the corresponding orthogonal projection P_U is symmetric we can write

$$P_U = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix},$$

with $A^* = A$, $B^* = B$. Explicitly $A = P_V \circ S|_V$, $B = P_{V^\perp} \circ S|_V$ and $C = P_{V^\perp} \circ S|_{V^\perp}$. From Equation (1.1) we get

$$\begin{pmatrix} \mathrm{Id}_V \\ Y \end{pmatrix} = P_U \begin{pmatrix} \mathrm{Id}_V \\ Y \end{pmatrix} = \begin{pmatrix} A + B^*Y \\ B + CY \end{pmatrix}, \quad 0 = P_U \begin{pmatrix} -Y^* \\ \mathrm{Id}_{V^\perp} \end{pmatrix} = \begin{pmatrix} -AY^* + B^* \\ -BY^* + C \end{pmatrix}.$$

In particular, $Y^* = A^{-1}B^*$ and, since A is self-adjoint,

$$Y = BA^{-1} \tag{1.2}$$

If we plug this relation into the equation $\mathrm{Id}_V = A + B^*Y$ we get

$$\mathrm{Id}_V = A(\mathrm{Id}_V + Y^*Y)$$

Since $\langle Y^*Yx, x\rangle = \langle Yx, Yx\rangle \geq 0$ the map $\mathrm{Id}_V + Y^*Y$ is always invertible.
This yields

$$A = (\mathrm{Id}_V + Y^*Y)^{-1}$$

In particular, $P_U \in W_V \cap \mathrm{G}_k(\mathbb{R}^n)$. Further, since $AY^* = B^*$, we get that

$$B = Y(\mathrm{Id}_V + Y^*Y)^{-1}$$

and, together with $C = BY^*$ we yield

$$C = Y(\mathrm{Id}_V + Y^*Y)^{-1}Y^*$$

Hence

$$P_U = \begin{pmatrix} (\mathrm{Id}_V + Y^*Y)^{-1} & (\mathrm{Id}_V + Y^*Y)^{-1}Y^* \\ Y(\mathrm{Id}_V + Y^*Y)^{-1} & Y(\mathrm{Id}_V + Y^*Y)^{-1}Y^* \end{pmatrix} \in W_V \cap \mathrm{G}_k(\mathbb{R}^n). \tag{1.3}$$

Equation (1.3) actually defines a smooth map

$$\phi\colon \mathrm{Hom}(V, V^\perp) \to W_V \cap \mathrm{G}_k(\mathbb{R}^n)$$

with left inverse given by Equation (1.2), which is smooth on W_V, hence ϕ is surjective and has full rank. Thus $\mathrm{G}_k(\mathbb{R}^n)$ is locally parametrized by $\mathrm{Hom}(V, V^\perp) \cong \mathbb{R}^{k\cdot(n-k)}$.

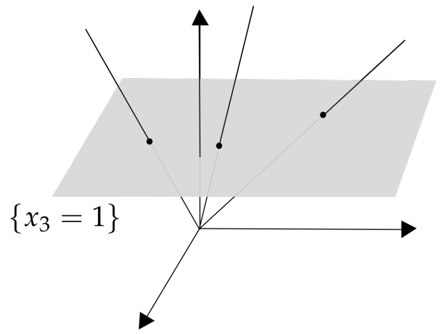


Figure 1.2: One possibility to visually identify $G_1(\mathbb{R}^3)$ and $\mathbb{RP}^2$.

Theorem 1.34. *The Grassmannian $G_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ (represented by the orthogonal projection onto these subspaces) is a submanifold of dimension $k(n-k)$.*

Exercise 1.35.
Show that $G_1(\mathbb{R}^3) \subset \mathrm{Sym}(3)$ is diffeomorphic to $\mathbb{RP}^2$.

Exercise 1.36 (Möbius band)**.**
Show that the Möbius band (without boundary)

$$M = \left\{ \left((2 + r\cos\tfrac{\varphi}{2})\cos\varphi, (2 + r\cos\tfrac{\varphi}{2})\sin\varphi, r\sin\tfrac{\varphi}{2} \right) \mid r \in (-\tfrac{1}{2}, \tfrac{1}{2}),\ \varphi \in \mathbb{R} \right\}$$

is a submanifold of $\mathbb{R}^3$. Show further that for each point $p \in \mathbb{RP}^2$ the open set $\mathbb{RP}^2 \setminus \{p\} \subset \mathbb{RP}^2$ is diffeomorphic to M.

1.4 Tangent Spaces in $\mathbb{R}^k$

Definition 1.37. *Let $M \subset \mathbb{R}^k$ be an n-dimensional submanifold and $p \in M$. Then $X \in \mathbb{R}^k$ is called a tangent vector of M at p if there is a smooth map $\gamma\colon (-\epsilon, \epsilon) \to M \subset \mathbb{R}^k$ such that $\gamma(0) = p$ and $\gamma'(0) = X$.*

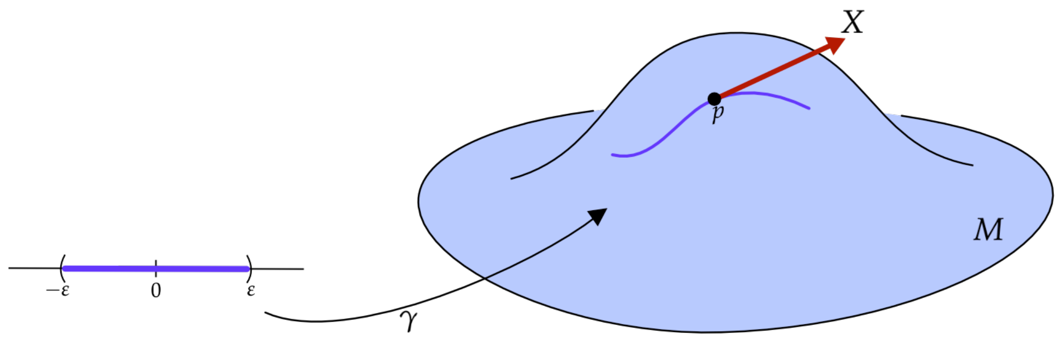


Figure 1.3: A tangent vector of M at p together with a curve γ.

Remark 1.38. Note that the smoothness of the map γ in the above definition is understood as smooth map to $\mathbb{R}^k$.

Definition 1.39. *The set of tangent vectors of M at a point p is called the tangent space T_pM of M at p.*

Theorem 1.40. *If $M \subset \mathbb{R}^k$ is an n-dimensional submanifold and $p \in M$, then T_pM is an n-dimensional linear subspace of $\mathbb{R}^k$.*

Proof. We male use of d) the equivalent definitions of submanifols of theorem 1.32 to obtain an open set $U \subset \mathbb{R}^n$, $q \in U$ and a smooth map $\psi\colon U \to M \subset \mathbb{R}^k$ such that $\psi(q) = p$ and $\dim \psi'(q)\mathbb{R}^n = \operatorname{rank} \psi'(q) = n$. Here

$$\psi'(q)\mathbb{R}^n := \{\psi'(q)X \mid X \in \mathbb{R}^n\}.$$

For $y \in \mathbb{R}^n$ define a curve as follows: Choose $\epsilon > 0$ small enough such that $q + tY \in U$ for $t \in (-\epsilon, \epsilon)$ and define

$$\gamma\colon (-\epsilon, \epsilon) \to M,\ t \mapsto \gamma(t) := \psi(q + tY).$$

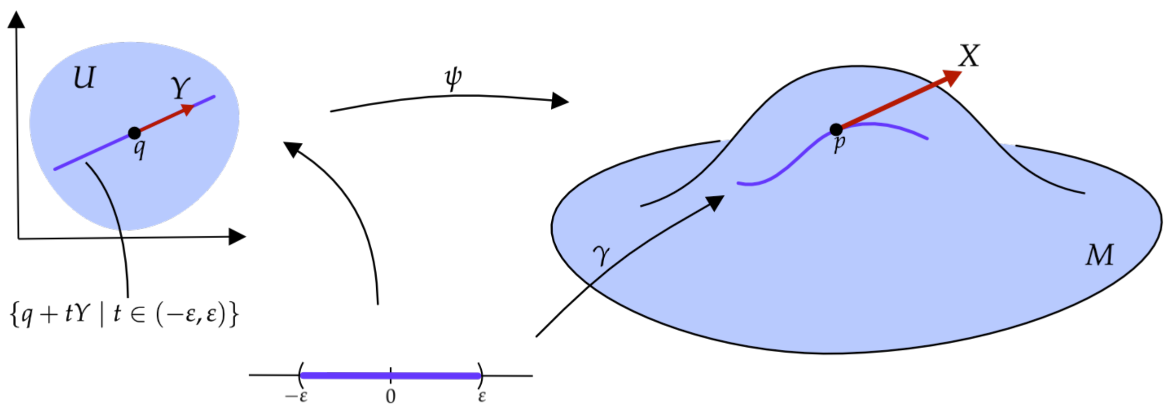


Figure 1.4: Construction of γ by mapping a straight line segment in U.

By the chain rule: $\gamma'(0) = \psi'(q)Y$, hence by definition $\psi'(q)Y = \gamma'(0) \in T_pM$. As Y was arbitrary we yield

$$\psi'(q)\mathbb{R}^n = \{\psi'(q)X \mid X \in \mathbb{R}^n\} \subset T_pM.$$

We note that $\{\psi'(q)X \mid X \in \mathbb{R}^n\}$ is a linear subspace of $\mathbb{R}^k$ as $\psi'(q)$ has full rank.
Now we use c) of theorem 1.32 to find an open set $W \subset \mathbb{R}^k$, $p \in W$ and a smooth map $g\colon W \to \mathbb{R}^{k-n}$ such that $g(x) = 0$ for all $x \in W \cap M$ and $g'(p)$ has rank $k - n$. If we now take $X \in T_pM$ it comes with a curve $\gamma\colon (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X$. Then with $g \circ \gamma(0) = g(p) = 0$ and the chain rule:

$$0 = g'(p) = g'(\gamma(0))\gamma'(0) = g'(\gamma(0))X,$$

so that $X \in \ker g'(p)$. As $\ker g'(p)$ is a linear subspace of $\mathbb{R}^k$ with dimension n we en up with

$$\operatorname{im} \psi'(q) \subset T_pM \subset \ker g'(p).$$

As the left and the right hand side of this inclusion are linear subspaces of the same dimension, $\operatorname{im} \psi'(q) \subset \ker g'(p)$ implies that

$$\operatorname{im} \psi'(q) = T_pM = \ker g'(p).$$

□

1.5 Matrix Lie Groups

Although we take, for our purposes, an "efficient" approach to Lie Groups by considering subgroups of $\mathrm{GL}(n,\mathbb{R})$, we also want to give the general definition:

Definition 1.41. *A set G with a map*

$$*\colon G\times G\to G,\ (x,y)\mapsto x*y := xy$$

is called a group if

1. $(xy)z = x(yz)$ *for all* $x,y,z\in G$.
2. *There is* $e\in G$ *such that for all* $x\in G$ *we have* $ex = xe = x$.
3. *For every* $x\in G$ *there is* $x^{-1}\in G$ *such that* $x^{-1}x = xx^{-1} = e$.

Definition 1.42. *A Lie group is a group G which is also a manifold such that the maps*

$$*\colon G\times G\to G,\ (x,y)\to xy$$
$$(\,)^{-1}\colon G\to G,\ x\mapsto x^{-1}$$

are smooth.

The most important example for us will be *general linear group*

$$\mathrm{Gl}(n,\mathbb{R}) := \{A\in\mathbb{R}^{n\times n}\mid \det A\neq 0\}$$

is an open set in $\mathbb{R}^{n\times n}$ and therefore a manifold. The group multiplication is matrix multiplication and therefore smooth. The same holds for $A\mapsto A^{-1}$, it can for instance be explicitly computet by Cramer's rule. Note that, as an open set of $\mathbb{R}^{n\times n}\cong\mathbb{R}^{n^2}$, we only need a single coordinate chart.

Definition 1.43. *A Lie subgroup of a lie group G is a submanifold $H\subset G$ which is also a subgroup of G:*

$$x,y\in H \implies xy\in H$$
$$x\in H \implies x^{-1}\in H$$

Exercise 1.44. Check that a Lie subgroup naturally has the structure of a Lie-group itself.

Definition 1.45. *A Lie subgroup of* $\mathrm{GL}(n,\mathbb{R})$ *is called a matrix Lie group.*

Example 1.46.

1. The orthogonal group $\mathrm{O}(n)$ is a matrix Lie group.

2. $\mathrm{SO}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det A = 1\}$ is the “special linear group”. Clearly this is a subgroup of $\mathrm{GL}(n,\mathbb{R})$. By defining

$$g\colon \mathbb{R}^{n\times n} \to \mathbb{R},\ A \mapsto \det A - 1$$

we see that $\mathrm{SO}(n,\mathbb{R})$ is the zeroset of g. It remains to check that

$$g(A) = 0 \implies g'(A) \neq 0.$$

Then part c) of theorem 1.32 yields the claim. For $A \in \mathrm{SO}(n,\mathbb{R})$ and $Y \in \mathbb{R}^{n\times n}$ choose $B\colon (-\epsilon,\epsilon) \to \mathbb{R}^{n\times n}$ with $B(0) = A$ and $B'(0) = Y$. for example $B(t) = a + tY$. Then, with $A = (a_1,\ldots,a_n)$ and $B = (b_1,\ldots,b_n)$ by definition of the directional derivative and the product rule

$$\begin{aligned} g'(A)(X) &= \frac{d}{dt}\bigg|_{t=0} g(B(t)) \\ &= \frac{d}{dt}\bigg|_{t=0} [\det B(t) - 1] \\ &= \frac{d}{dt}\bigg|_{t=0} [\det(b_1(t),\ldots,b_n(t)) - 1] \\ &= \det\big(b_1'(0), a_2,\ldots,a_n\big) + \ldots + \det\big(a_1,\ldots,a_{n-1}, b_n'(0)\big) \end{aligned}$$

As the colums of A are by assumption linearly independent we can write for every $j \in \{1,\ldots,n\}$

$$b_j'(0) = \sum_{k=1}^{n} x_{kj} a_k = \sum_{k=1}^{n} a_k x_{kj}.$$

Writing this in matrix form gives

$$B'(0) = AX$$

where

$$X = \begin{pmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{pmatrix},$$

in other words: $X = A^{-1}B'(0)$. Hence

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} g(B(t)) &= \det\left(\sum_{k=1}^{n} x_{k1} a_k, a_2, \ldots, a_n\right) + \\ &\ \ \vdots \\ &\quad + \det\left(a_1,\ldots,a_{n-1}, \sum_{k=1}^{n} x_{kn} a_k\right) \\ &= \operatorname{tr}(X)\det(A) \end{aligned}$$

Choose now any $X \in \mathbb{R}^{n\times n}$ with $\operatorname{tr}(X) \neq 0$ and define $Y := AX$, then

$$g'(A)(Y) = \operatorname{tr}(X) \neq 0.$$

Therefore $\mathrm{SL}(n,\mathbb{R})$ is an (n^2-1)-dimensional submanifold of $\mathbb{R}^{n\times n}$.

3. The "special orthogonal group"

$$\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$$

is a matrix Lie group of dimension $\frac{n(n-1)}{2}$.

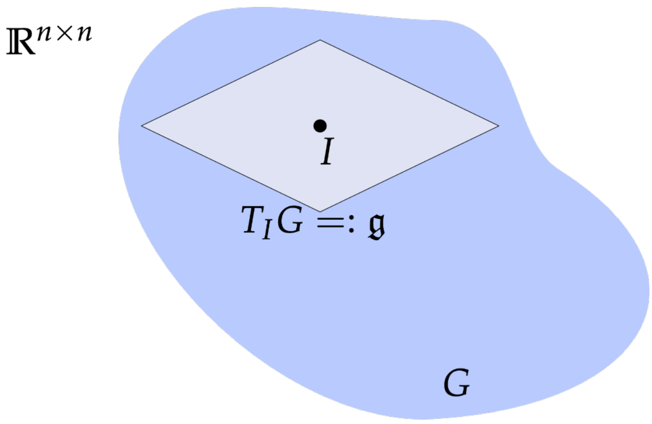


Figure 1.5: The tangent space $T_I G$ at the identity is the Lie algebra.

The tangent space of a Lie group G is commonly written as the same letters in fracutred font $\mathfrak{g}$.

Definition 1.47. *If $G \subset \mathbb{R}^{n\times n}$ is a matrix Lie group, then then tangent space $\mathfrak{g} := T_i G$ is called the Lie algebra of G.*

Remark 1.48. We've already shown that the Lie algebra is a vector space.

Definition 1.49. *A Lie algebra is a real vector space $\mathfrak{g}$ togehter with a bilinear map*

$$[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that for all $X, Y, Z \in \mathfrak{g}$:

1. $[X, Y] = -[Y, X]$ *(skew-symmetry).*
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jacobi-identity).*

Example 1.50. Consider $\mathfrak{g} = \mathbb{R}^{n\times n} = T_I \mathrm{GL}(n, \mathbb{R})$ then

$$[X, Y] := XY - YX.$$

Then obviously

$$[X, Y] = -[Y, X]$$

and

$$[X, [Y, Z]] = X(YZ - ZY) - (YZ - ZY)X = XYZ - XZY - YZX + ZYX$$

so that adding the cyclic permutations yields the Jacobi-identity.

Definition 1.51. *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ is a linear subspace. Then $\mathfrak{h}$ is called a Lie subalgebra if*

$$X, Y \in \mathfrak{h} \implies [X, Y] \in \mathfrak{h}.$$

Lemma 1.52. *$B\colon (a, b) \to \mathrm{GL}(\mathbb{R})$ smooth, then for all $t \in (a, b)$*

$$\left(B^{-1}\right)' = -B^{-1}B'B^{-1}.$$

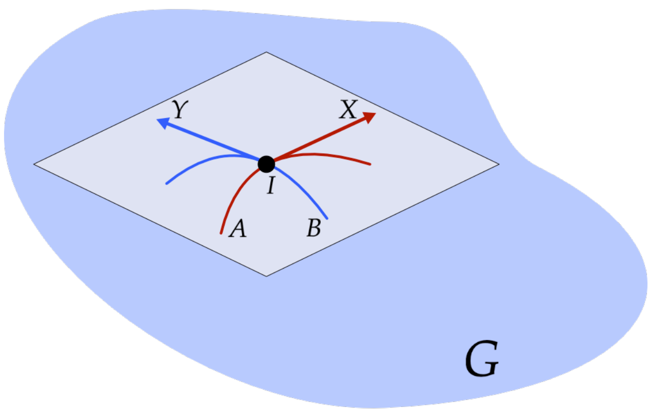
Proof. We differentiate the equality $B^{-1}B = I$:

$$(B^{-1})'B + B^{-1}B' = 0 \quad \Leftrightarrow \quad (B^{-1})' = -B^{-1}B'B^{-1}.$$

□

Theorem 1.53. *If $G \subset \mathrm{GL}(n, \mathbb{R})$ is a matrix Lie group, then $\mathfrak{g} = T_I G$ is a Lie subalgebra of $\mathbb{R}^{n \times n}$ with the standard cummutator as defined above.*

Proof. Let $X, Y \in T_I G$. We have to show that $XY - YX \in T_I G$. By the definition of tangent vectors we can choose $A, B\colon (-\epsilon, \epsilon) \to G$ auch that $A(0) = B(0) = I$ and $A'(0) = X$ and $B'(0) = Y$.



We first show that for all $t \in (-\epsilon, \epsilon)$ we have that

$$B(t)^{-1}XB(t) \in T_I G = \mathfrak{g}.$$

To this end define $C\colon (-\epsilon, \epsilon) \to G$ by

$$C(s) := B^{-1}(t)A(s)B(t).$$

Then

$$C(0) = B^{-1}(t)IB(t) = I,$$

hence (as $B(t)$ is constant in s)

$$C'(0) = B^{-1}(t)XB(t).$$

Therefore

$$B^{-1}(t)XB(t) \in T_I G = \mathfrak{g}$$

for all $t \in (-\epsilon, \epsilon)$.

Moreover,

$$\begin{aligned}\mathfrak{g} \ni \left.\frac{d}{dt}\right|_{t=0} \left(B^{-1}(t)XB(t)\right) &= (-B^{-1}(0)B'(0)B^{-1}(0))XB(0) + B^{-1}(0)XB'(0) \\ &= -YX + XY \\ &= [X, Y].\end{aligned}$$

□

Example 1.54. For $G = \mathrm{SO}(n)$ it is

$$T_I G = \mathfrak{so}(n) = \{X \in \mathbb{R}^{n\times n} \mid X^T = -X\}.$$

Then for $X, Y \in \mathfrak{so}(n)$ we have

$$[X, Y]^T = (XY - YX)^T = Y^T X^T - X^T Y^T = YX - XY = -[X, Y]$$

hence $[X, Y] \in \mathfrak{so}(n)$.

2. The Tangent Bundle

2.1 Tangent Vectors

Let M be an n-dimensional smooth manifold. We will define for each $p \in \mathrm{M}$ an n-dimensional vector space $\mathrm{T}_p\mathrm{M}$, the *tangent space of* M *at* p.

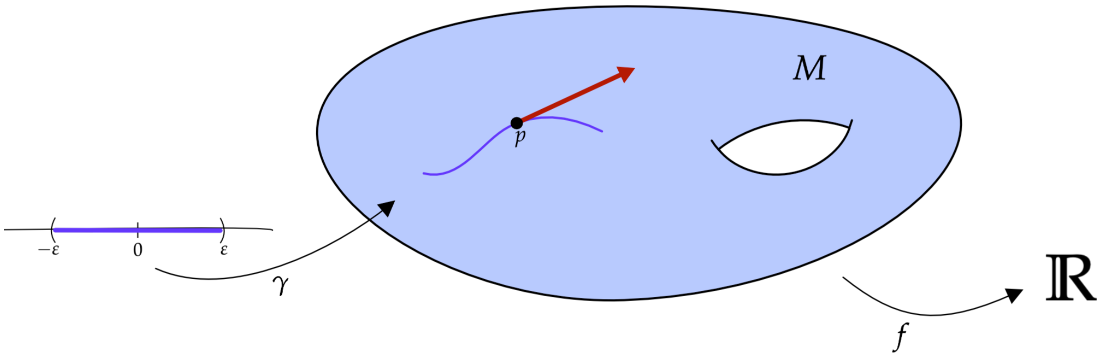
Definition 2.1 (Tangent space). *Let* M *be a smooth n-manifold and* $p \in \mathrm{M}$. *A tangent vector* X *at* p *is then a linear map*

$$X \colon \mathscr{C}^\infty(\mathrm{M}) \to \mathbb{R}, \quad f \mapsto Xf$$

such that there is a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to \mathrm{M}$ *with* $\gamma(0) = p$ *and*

$$Xf = (f \circ \gamma)'(0).$$

The tangent space is then the set of all tangent vectors $\mathrm{T}_p\mathrm{M} := \{X \mid X \text{ tangent vector at } p\}$.



Let $\varphi = (x_1, \ldots, x_n)$ be a chart defined on $U \ni p$. Let $\tilde{f} = f \circ \varphi^{-1}$, $\tilde{\gamma} = \varphi \circ \gamma$ and $\tilde{p} = \varphi(p)$. Then

$$Xf = (f \circ \gamma)'(0) = (\tilde{f} \circ \tilde{\gamma})'(0) = \left(\partial_1 \tilde{f}(\tilde{p}), \ldots, \partial_n \tilde{f}(\tilde{p})\right) \begin{pmatrix} \tilde{\gamma}_1'(0) \\ \vdots \\ \tilde{\gamma}_n'(0) \end{pmatrix}.$$

So tangent vectors can be parametrized by n numbers $\alpha_i = \tilde{\gamma}_i'(0)$:

$$Xf = \alpha_1 \partial_1 \tilde{f}(\tilde{p}) + \cdots + \alpha_n \partial_n \tilde{f}(\tilde{p}).$$

Exercise 2.2.
Within the setup above, show that to each vector $\alpha \in \mathbb{R}^n$, there exists a curve $\gamma \colon (-\varepsilon, \varepsilon) \to \mathrm{M}$ such that $\gamma(0) = p$ and $(f \circ \gamma)'(0) = \alpha_1 \partial_1 \tilde{f}(\tilde{p}) + \cdots + \alpha_n \partial_n \tilde{f}(\tilde{p})$.

Definition 2.3 (Coordinate frame). *If $\varphi = (x_1, \dots, x_n)$ is a chart at $p \in \mathrm{M}$, $f \in \mathscr{C}^\infty(\mathrm{M})$. Then*

$$\left.\frac{\partial}{\partial x_i}\right|_p f := \partial_i (f \circ \varphi^{-1})(\varphi(p)), \quad i = 1, \dots, n$$

is called a coordinate frame of T_pM at p.

Interlude: How to construct $\mathscr{C}^\infty$ functions on the whole of M?

Toolbox: $f\colon \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ e^{-1/x} & \text{for } x > 0. \end{cases}$$

is $\mathscr{C}^\infty$ and so is then $g(x) = f(1 - x^2)$ and $h(x) = \int_0^x g$. From h this we can build a smooth function $\hat{h}\colon \mathbb{R} \to [0,1]$ with $\hat{h}(x) = 1$ for $x \in [-\frac{1}{4}, \frac{1}{4}]$ and $\hat{h}(x) = 0$ for $x \in \mathbb{R} \setminus (-1,1)$. Then we can define a smooth function $\tilde{h}\colon \mathbb{R}^n \to \mathbb{R}$ by $\tilde{h}(x) = \hat{h}(x_1^2 + \cdots x_n^2)$ which vanishes outside the unit ball and is constant $= 1$ inside the ball of radius $\frac{1}{2}$.

Theorem 2.4. *Let* M *be a smooth n-manifold, $p \in \mathrm{M}$ and (U, φ) a chart with $U \ni p$. Let $a_1, \dots, a_n \in \mathbb{R}$. Then there is $f \in \mathscr{C}^\infty(\mathrm{M})$ such that*

$$\left.\frac{\partial}{\partial x_i}\right|_p f = a_i, \quad i = 1, \dots, n.$$

Proof. We define $\tilde{g}\colon \mathbb{R}^n \to \mathbb{R}$, $\tilde{g}(x) = \tilde{h}(\lambda(x - \varphi(p)))$ with λ such that $\tilde{g}(x) = 0$ for all $x \notin \varphi(U)$. Then let

$$\tilde{f}\colon \mathbb{R}^n \to \mathbb{R},\ \tilde{f}(x) := \tilde{g}(x)(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$$

Then $f\colon \mathrm{M} \to \mathbb{R}$ given by

$$f(q) = \begin{cases} \tilde{f}(\varphi(q)) & \text{for } q \in U, \\ 0 & \text{for } q \notin U \end{cases}$$

is such a function. □

Corollary 2.5. *The set of vectors $\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$ is linearly independent.*

Corollary 2.6. $\mathrm{T}_p\mathrm{M} \subset \mathscr{C}^\infty(\mathrm{M})^*$ *is an n-dimensional linear subspace.*

Proof. Follows from the last corollary and from Exercise 2.2, which shows that $\mathrm{T}_p\mathrm{M}$ is a subspace spanned by $\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$. □

Theorem 2.7 (Transformation of coordinate frames). *If (U, φ) and (V, ψ) are charts with $p \in U \cap V$, $\varphi|_{U \cap V} = \Phi \circ \psi|_{U \cap V}$. Then for every $X \in \mathrm{T}_p\mathrm{M}$,*

$$X = \sum a_i \left.\frac{\partial}{\partial x_i}\right|_p = \sum b_i \left.\frac{\partial}{\partial y_i}\right|_p ,$$

where $\varphi = (x_1, \ldots, x_n)$, $\psi = (y_1, \ldots, y_n)$, we have

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \Phi'(\psi(p)) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Proof. Let $\gamma \colon (-\varepsilon, \varepsilon) \to \mathrm{M}$ such that $Xf = (f \circ \gamma)'(0)$. Let $\tilde{\gamma} = \varphi \circ \gamma$ and $\hat{\gamma} = \psi \circ \gamma$, then

$$a = \tilde{\gamma}'(0), \quad b = \hat{\gamma}'(0).$$

Let $\Phi \colon \psi(U \cap V) \to \varphi(U \cap V)$ be the coordinate change $\Phi = \varphi \circ \psi^{-1}$. Then

$$\tilde{\gamma} = \varphi \circ \gamma = \Phi \circ \psi \circ \gamma = \Phi \circ \hat{\gamma}.$$

In particular,

$$a = \tilde{\gamma}'(0) = (\Phi \circ \hat{\gamma})'(0) = \Phi'(\psi(p))\hat{\gamma}'(0) = \Phi'(\psi(p))b.$$

□

Definition 2.8. *Let M and $\tilde{\mathrm{M}}$ be smooth manifolds, $f \colon \mathrm{M} \to \tilde{\mathrm{M}}$ smooth, $p \in \mathrm{M}$. Then define a linear map $d_p f \colon \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\tilde{\mathrm{M}}$ by setting for $g \in \mathscr{C}^\infty(\tilde{\mathrm{M}})$ and $X \in \mathrm{T}_p\mathrm{M}$*

$$d_p f(X)g := X(g \circ f).$$

Remark 2.9. $d_p f(X)$ is really a tangent vector in $\mathrm{T}_p\tilde{\mathrm{M}}$ because, if X corresponds to a curve $\gamma \colon (-\varepsilon, \varepsilon) \to \mathrm{M}$ with $\gamma(0) = p$ then

$$d_p f(X)g = \left.\frac{d}{dt}\right|_{t=0} (g \circ f) \circ \gamma = \left.\frac{d}{dt}\right|_{t=0} g \circ \underbrace{(f \circ \gamma)}_{=:\tilde{\gamma}} = \left.\frac{d}{dt}\right|_{t=0} g \circ \tilde{\gamma}.$$

Notation:

The tangent vector $X \in \mathrm{T}_p\mathrm{M}$ corresponding to a curve $\gamma \colon (-\varepsilon, \varepsilon) \to \mathrm{M}$ with $\gamma(0) = p$ is denoted by $X =: \gamma'(0)$.

Theorem 2.10 (Chain rule). *Suppose $g \colon \mathrm{M} \to \tilde{\mathrm{M}}$, $f \colon \tilde{\mathrm{M}} \to \hat{\mathrm{M}}$ are smooth maps, then*

$$d_p(f \circ g) = d_{g(p)} f \circ d_p g.$$

Definition 2.11 (Tangent bundle)**.** *The set*

$$\mathrm{TM} := \bigsqcup_{p\in \mathrm{M}} \mathrm{T}_p\mathrm{M}$$

is called the tangent bundle of M*. The map*

$$\pi\colon \mathrm{TM} \to \mathrm{M}, \mathrm{T}_p\mathrm{M} \ni X \to p$$

is called the projection map. So $\mathrm{T}_p\mathrm{M} = \pi^{-1}(\{p\})$*.*

Most elegant version of the chain rule:

If $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ is smooth, then $df\colon \mathrm{T}M \to T\tilde{\mathrm{M}}$ where $df(X) = d_{\pi(X)}f(X)$. With this notation,

$$d(f\circ g) = df \circ dg.$$

Theorem 2.12. *If* $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ *is a diffeomorphism then for each* $p \in \mathrm{M}$ *the map*

$$d_pf\colon \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\tilde{\mathrm{M}}$$

is a vector space isomorphism.

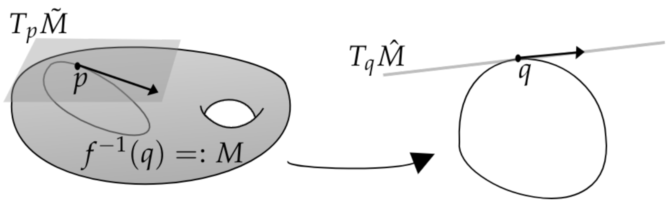
Proof. f is bijective and f^{-1} is smooth, $\mathrm{Id}_\mathrm{M} = f^{-1}\circ f$. For all $p \in \mathrm{M}$,

$$\mathrm{Id}_{\mathrm{T}_p\mathrm{M}} = d_p(\mathrm{Id}_\mathrm{M}) = d_{f(p)}f^{-1}\circ d_pf.$$

So d_pf is invertible. □

Theorem 2.13 (Manifold version of the inverse function theorem)**.**
Let $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ *be smooth,* $p \in \mathrm{M}$ *with* $d_pf\colon \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\tilde{\mathrm{M}}$ *invertible. Then there are open neighborhoods* $U \subset \mathrm{M}$ *of* p *and* $V \subset \tilde{\mathrm{M}}$ *of* $f(p)$ *such that* $f|_U : U \to V$ *is a diffeomorphism.*

Proof. The theorem is a reformulation of the inverse function theorem. □



Theorem 2.14 (Submersion theorem)**.** *Let* $f\colon \tilde{\mathrm{M}} \to \hat{\mathrm{M}}$ *be a submersion, i.e. for each* $p \in \tilde{\mathrm{M}}$ *the derivative* $d_pf\colon \mathrm{T}_p\tilde{\mathrm{M}} \to \mathrm{T}_{f(p)}\hat{\mathrm{M}}$ *is surjective. Let* $q = f(p)$ *be fixed. Then*

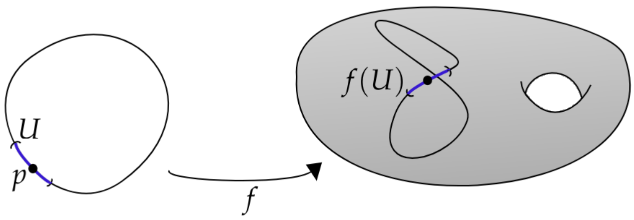
$$\mathrm{M} := f^{-1}(\{q\})$$

is an n-dimensional submanifold of $\tilde{\mathrm{M}}$*, where* $n = \dim\tilde{\mathrm{M}} - \dim\hat{\mathrm{M}}$*.*

Remark 2.15. The sumbersion theorem is a manifold version of the implicit function theorem.

Proof. Take charts and apply Theorem 1.32. □

Theorem 2.16 (Immersion theorem)**.** *Let $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ be an immersion, i.e. for every $p \in \mathrm{M}$ the differential $d_p f\colon \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\tilde{\mathrm{M}}$ is injective. Then for each $p \in \mathrm{M}$ there is an open set $U \subset \mathrm{M}$ with $U \ni p$ such that $f(U)$ is a submanifold of $\tilde{\mathrm{M}}$.*



Proof. Take charts and apply Theorem 1.32. □

Is there a global version, i.e. without passing to $U \subset \mathrm{M}$? Assuming that f is injective is not enough.

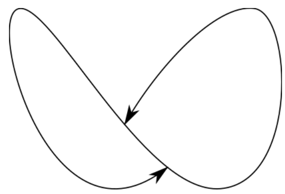


Figure 2.1: Example of an injective immersion that is no submanifold.

Exercise 2.17.
Let $f\colon \mathrm{N} \to \mathrm{M}$ be a smooth immersion. Prove: If f is moreover a *topological embedding*, i.e. its restriction $f\colon \mathrm{N} \to f(\mathrm{N})$ is a homeomorphism between N and $f(\mathrm{N})$ (with its subspace topology), then $f(\mathrm{N})$ is a smooth submanifold of M.

Exercise 2.18.
Let M be compact, $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ an injective immersion, then $f(\mathrm{M})$ is a submanifold.

Exercise 2.19.
Let $\mathrm{X} := \mathbb{C}^2 \setminus \{0\}$. The complex projective plane is the quotient space $\mathbb{C}\mathrm{P}^1 = X/_{\sim}$, where the equivalence relation is given by

$$\psi \sim \tilde{\psi} :\Leftrightarrow \lambda\psi = \tilde{\psi}, \quad \lambda \in \mathbb{C}.$$

Consider $\mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$, then the *Hopf fibration* is the map

$$\pi\colon \mathbb{S}^3 \to \mathbb{C}\mathrm{P}^1, \quad \psi \mapsto [\psi].$$

Show: For each $p \in \mathbb{C}\mathrm{P}^1$ the *fiber* $\pi^{-1}(\{p\})$ is a submanifold diffeomorphic to $\mathbb{S}^1$.

2.2 The tangent bundle as a smooth vector bundle

Let M be a smooth n-manifold, $p \in$ M. The tangent space at p is an n-dimensional subspace of $(\mathscr{C}^\infty(\mathrm{M}))^*$ given by

$\mathrm{T}_p\mathrm{M} = \{X \in (\mathscr{C}^\infty(\mathrm{M}))^* \mid \exists \gamma\colon (-\varepsilon, \varepsilon) \to \mathrm{M}, \gamma(0) = p, Xf = (f \circ \gamma)'(0) = Xf, \forall f \in \mathscr{C}^\infty(\mathrm{M})\}$

The *tangent bundle* is then the set

$$\mathrm{TM} = \bigsqcup_{p \in \mathrm{M}} \mathrm{T}_p\mathrm{M}$$

and comes with a projection

$$\pi\colon \mathrm{TM} \to \mathrm{M}, \qquad \mathrm{T}_p\mathrm{M} \ni X \mapsto p \in \mathrm{M}.$$

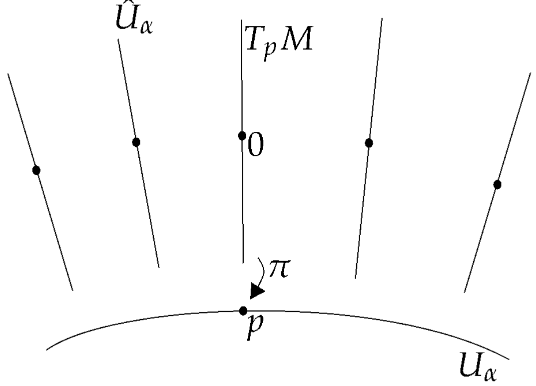
The set $\pi^{-1}(\{p\}) = \mathrm{T}_p\mathrm{M}$ is called the *fiber of the tangent bundle at* p.

Goal: We want to make TM into a $2n$-dimensional manifold.

If $\varphi = (x_1, \dots, x_n)$ be a chart of M defined on $U \ni p$. Then we have a basis $\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p$ of $\mathrm{T}_p\mathrm{M}$. So there are unique $y_1(X), \dots, y_n(X) \in \mathbb{R}$ such that

$$X = \sum y_i(X) \left.\frac{\partial}{\partial x_i}\right|_p.$$

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a smooth atlas of M. For each $\alpha \in A$ we get an open set $\hat{U}_\alpha := \pi^{-1}(U_\alpha)$ and a function $y_\alpha\colon \hat{U}_\alpha \to \mathbb{R}^n$ which maps a given vector to the coordinates $y_\alpha = (y_{\alpha,1}, \dots, y_{\alpha,n})$ with respect to the frame defined by φ_α.



Now, we define

$$\hat{\varphi}_\alpha\colon \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

by

$$\hat{\varphi}_\alpha = (\varphi_\alpha \circ \pi, y_\alpha).$$

For any two charts we have a transition map $\phi_{\alpha\beta}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ such that $\varphi_\beta|_{U_\alpha \cap U_\beta} = \phi_{\alpha\beta} \circ \varphi_\alpha|_{U_\alpha \cap U_\beta}$. The chain rule yields:

$$y_\beta(X) = \phi'_{\alpha\beta}(\varphi_\alpha(\pi(X)))y_\alpha(X).$$

Hence we see that $\hat{\varphi}_\beta \circ \hat{\varphi}_\alpha^{-1}$ is a diffeomorphism.
Topology on TM**:**

$$\mathcal{O}_{\mathrm{TM}} := \left\{W \subset \mathrm{TM} \mid \hat{\varphi}_\alpha(W \cap \hat{U}_\alpha) \in \mathcal{O}_{\mathbb{R}^{2n}} \textit{ for all } \alpha \in A\right\}.$$

Exercise 2.20.

a) This defines a topology on TM.

b) With this topology TM is Hausdorff and 2nd-countable.

c) All $\hat{\varphi}_\alpha$ are homeomrophisms onto their image.

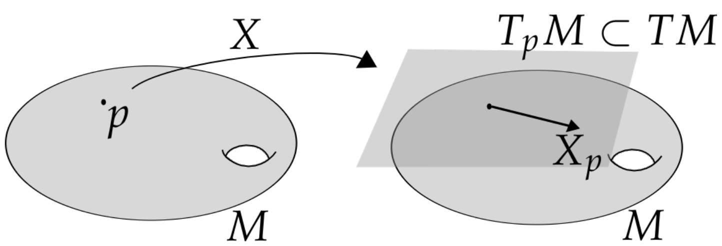
Because coordinate changes are smooth, this turns TM into a smooth $2n$-dimensional manifold.

Definition 2.21 (Vector field). *A (smooth) vector field on a manifold* M *is a smooth map*

$$X\colon \mathrm{M} \to \mathrm{TM}$$

with $\pi \circ X = \mathrm{Id}_\mathrm{M}$ *i.e.* $X(p) \in \mathrm{T}_p\mathrm{M}$ *for all* $p \in \mathrm{M}$.

Remark 2.22. Usually we write X_p instead of $X(p)$. If X is a vector field and $f \in \mathscr{C}^\infty(\mathrm{M})$, then $Xf \in \mathscr{C}^\infty(M)$ is given by $(Xf)(p) = X_p f$.
Read: "X differentiates f".



Exercise 2.23.
Show that each of the following conditions is equivalent to the smoothness of a vector field X as a section $X\colon \mathrm{M} \to \mathrm{TM}$:

a) For each $f \in \mathscr{C}^\infty(\mathrm{M})$, the function Xf is also smooth.

b) If we write $X|_U =: \sum v_i \frac{\partial}{\partial x_i}$ in a coordinate chart $\varphi = (x_1, \dots, x_n)$ defined on $U \subset \mathrm{M}$, then the components $v_i\colon U \to \mathbb{R}$ are smooth.

Exercise 2.24.
On $\mathbb{S}^2 = \{x = (x_0, x_1, x_2) \mid \|x\| = 1\} \subset \mathbb{R}^3$ we consider coordinates given by the stereographic projection from the north pole $N = (1, 0, 0)$:

$$y_1 = \tfrac{x_1}{1-x_0}, \quad y_2 = \tfrac{x_2}{1-x_0}.$$

Let the vector fields X and Y on $\mathbb{S}^2 \setminus \{N\}$ be defined in these coordinates by

$$X = y_2 \tfrac{\partial}{\partial y_1} - y_1 \tfrac{\partial}{\partial y_2}, \qquad Y = y_1 \tfrac{\partial}{\partial y_1} + y_2 \tfrac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the stereographic projection from the south pole $S = (-1, 0, 0)$.

Exercise 2.25.
Prove that the tangent bundle of a product of smooth manifolds is diffeomorphic to the product of the tangent bundles of the manifolds. Deduce that the tangent bundle of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$.

3. Vector bundles

Definition 3.1 (Vector bundle). *A smooth vector bundle of rank k is a triple* $(\mathrm{E}, \mathrm{M}, \pi)$ *which consists of smooth manifolds* E *and* M *and a smooth map*

$$\pi \colon \mathrm{E} \to \mathrm{M}$$

such that for each $p \in \mathrm{M}$

(i) the fiber $\mathrm{E}_p := \pi^{-1}(\{p\})$ *has the structure of a k-dimensional vector space*

(ii) each $p \in \mathrm{M}$ *has an open neighborhood* $U \subset \mathrm{M}$ *such that there exists a diffeomorphism*

$$\phi \colon \pi^{-1}(U) \to U \times \mathbb{R}^k$$

such that $\pi_U \circ \phi = \pi$ *and for each* $p \in \mathrm{M}$ *the restriction* $\pi_{\mathbb{R}^k} \circ \phi|_{\mathrm{E}_p}$ *is a vector space isomorphism.*

Definition 3.2 (Section). *Let* E *be a smooth vector bundle over* M. *A section of* E *is a smooth map* $\psi \colon \mathrm{M} \to \mathrm{E}$ *such that* $\pi \circ \psi = \mathrm{Id}_{\mathrm{M}}$.

$$\Gamma(\mathrm{E}) := \{\psi \colon \mathrm{M} \to \mathrm{E} \mid \psi \text{ section of } \mathrm{E}\}$$

Example 3.3.

a) We have seen that the tangent bundle TM of a smooth manifold is a vector bundle of rank dim M. Its smooth sections were called vector fields. denoted by ΓTM

b) The product $\mathrm{M} \times \mathbb{R}^k$ is called the trivial bundle of rank k. Its smooth sections can be identified with $\mathbb{R}^k$-valued functions. More precisely, if $\pi_2 \colon \mathrm{M} \times \mathbb{R}^k \to \mathbb{R}^k$, then

$$\Gamma(\mathrm{M} \times \mathbb{R}^k) \ni \psi \longleftrightarrow f := \pi_2 \circ \psi \in \mathscr{C}^\infty(\mathrm{M}).$$

From now on we will keep this identification in mind.

Ways to make new vector bundles out of old ones

General principle: Any linear algebra operation that given new vector spaces out of given ones can be applied to vector bundles over the same base manifold.

Example 3.4.
Let E be a rank k vector bundle over M and F be a rank ℓ vector bundle over M.

a) Then $E \oplus F$ denotes the rank $k + \ell$ vector bundle over M the fibers of which are given by $(E \oplus F)_p = E_p \oplus F_p$.

b) Then $\mathrm{Hom}(E, F)$ denotes the rank $k \cdot \ell$ vector bundle over M with fiber given by $\mathrm{Hom}(E, F)_p := \{f \colon E_p \to F_p \mid f \text{ linear}\}$.

c) $E^* = \mathrm{Hom}(E, M \times \mathbb{R})$ with fibers $(E^*)_p = (E_p)^*$.

Let $E_1, \dots . E_r, F$ be vector bundles over M.

d) Then a there is new vector bundle $E_1^* \otimes \cdots \otimes E_r^* \otimes F$ of rank $\mathrm{rank}E_1 \cdots \mathrm{rank}E_r \cdot \mathrm{rank}F$ with fiber at p given by $E_{1p}^* \otimes \cdots \otimes E_{rp}^* \otimes F_p = \{\beta \colon E_{1p} \times \ldots \times E_{rp} \to F_p \mid \beta \text{ multilinear}\}$.

Exercise 3.5.
Give an explicit description of the (natural) bundle charts for the bundles (written down as sets) in the previous example.

Starting from TM:

a) $T^*M := (TM)^*$ is called the *cotangent bundle*.

b) Bundles of multilinear forms with all the $E_1, \dots, E_r, F$ copies of TM, T^*M or $M \times \mathbb{R}$ are called *tensor bundles*. Sections of such bundles are called *tensor fields*.

Example 3.6 (tautological bundle)**.**
We have seen that

$$G_k(\mathbb{R}^n) = \{\text{Orthogonal projections onto } k\text{-dim subspaces of } \mathbb{R}^n\}$$

is an $(n-k)k$-dimensional submanifold of $\mathrm{Sym}(n)$. Now, we can define the *tautological bundle* as follows:

$$E = \{(P, v) \in G_k(\mathbb{R}^n) \mid Pv = v\}.$$

W is an open neighborhood of P_V as described in the Grassmannian example. Then for $(P_U, v) \in E$ define $\phi(P_U, v) \in W \times V \cong W \times \mathbb{R}^k$ by $\phi(P_u, v) = (P_U, P_V v)$. Check that this defines a local trivialization.

Exercise 3.7.
Let $M \subset \mathbb{R}^k$ be a smooth submanifold of dimension n. Let $\iota \colon M \hookrightarrow \mathbb{R}^k$ denote the inclusion map. Show that the normal bundle $NM = \bigsqcup_{p \in M} (T_pM)^\perp \subset \iota^* T\mathbb{R}^k \cong M \times \mathbb{R}^k$ is a smooth rank $k - n$ vector bundle over M.

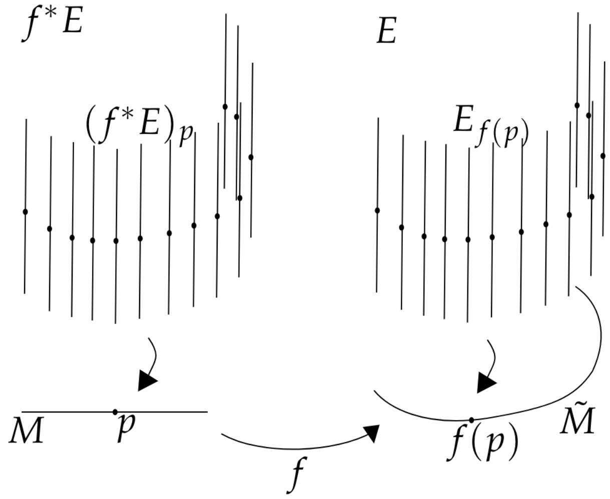
Definition 3.8 (pullback bundle)**.** *Given a smooth map* $f \colon M \to \tilde{M}$ *and a vector bundle* $E \to \tilde{M}$. *Then the pullback bundle* f^*E *is defined as the disjoint union of the fibers*

$$(f^*E)_p = E_{f(p)},$$

in other words

$$f^*E = \bigsqcup_{p \in M} E_{f(p)} \subset M \times E.$$

Exercise 3.9.
The set f^*E is a smooth submanifold of $M \times E$.



Definition 3.10 (Vector bundle isomorphism). *Two vector bundles* $\mathrm{E} \to \mathrm{M}, \tilde{\mathrm{E}} \to \mathrm{M}$ *are called isomorphic if there exists a bundle isomorphism between* E *and* $\tilde{\mathrm{E}}$*, i.e. a diffeomorphism*

$$f\colon \mathrm{E} \to \tilde{\mathrm{E}},$$

such that $\tilde{\pi} \circ f = \pi$ *(fibers to fibers) and* $f|_{\mathrm{E}_p} : \mathrm{E}_p \to \tilde{\mathrm{E}}_p$ *is a vector space isomorphism.*

Fact: (without proof)
Every rank k vector bundle E over M is isomorphic to $f^*\tilde{\mathrm{E}}$, where $\tilde{\mathrm{E}}$ is the tautological bundle over $\mathrm{G}_k(\mathbb{R}^n)$ (some n) and some smooth $f\colon \mathrm{M} \to \mathrm{G}_k(\mathbb{R}^n)$.

Definition 3.11 (trivial vector bundle). *A vector bundle* $\mathrm{E} \to \mathrm{M}$ *of rank k is called trivial if it is isomorphic to the trivial bundle* $\mathrm{M} \times \mathbb{R}^k$.

Remark 3.12. If $\mathrm{E} \to \mathrm{M}$ is a vector bundle of rank k then, by definition, each point $p \in \mathrm{M}$ has an open neighborhood U such that the restricted bundle $\mathrm{E}|_U := \pi^{-1}(U)$ is trivial, i.e. each bundle is locally trivial.

Definition 3.13 (Frame field). *Let* $\mathrm{E} \to M$ *be a rank k vector bundle,* $\varphi_1, \ldots, \varphi_k \in \Gamma(\mathrm{E})$*. Then* $(\varphi_1, \ldots, \varphi_k)$ *is called a frame field if for each* $p \in \mathrm{M}$ *the vectors* $\varphi_1(p), \ldots, \varphi_k(p) \in \mathrm{E}_p$ *form a basis.*

Proposition 3.14. E *is trivial if and only if* E *has a frame field.*

Proof.

"⇒": E trivial $\Rightarrow \exists F \in \Gamma\mathrm{Hom}(\mathrm{E}, \mathrm{M} \times \mathbb{R}^k)$ such that $F_p\colon \mathrm{E}_p \to \{p\} \times \mathbb{R}^k$ is a vector space isomorphism for each p. Then, for $i = 1, \ldots, k$ define $\varphi_i \in \Gamma(\mathrm{E})$ by $\varphi_p = F^{-1}(\{p\} \times e_i)$.

"⇐": $(\varphi_1, \ldots, \varphi_k)$ frame field $\rightsquigarrow$ define $F \in \Gamma\mathrm{Hom}(\mathrm{E}, \mathrm{M} \times \mathbb{R}^k)$ as the unique map such that $F_p(\varphi_i(p)) = \{p\} \times e_i$ for each $p \in \mathrm{M}$. $\rightsquigarrow$ F is a bundle isomorphism.

□

From the definition of a vector bundle: Each $p \in \mathrm{M}$ has a neighborhood U such that $\mathrm{E}|_U$ has a frame field.

Theorem 3.15. *For each $p \in \mathrm{M}$ there is an open neighborhood U and $\varphi_1, \ldots, \varphi_k \in \Gamma(\mathrm{E})$ such that $\varphi_1|_U, \ldots, \varphi_k|_U$ is a frame field of $\mathrm{E}|_U$.*

Proof. There is an open neighborhood $\tilde{U}$ of p such that $\mathrm{E}|_{\tilde{U}}$ is trivial. Thus there is a frame field $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_k \in \Gamma(\mathrm{E}|_{\tilde{U}})$. There is a subset $U \subset \tilde{U}$, a compact subset C with $U \subset C \subset \tilde{U}$ and a smooth function $f \in \mathscr{C}^\infty(\mathrm{M})$ such that $f|_U \equiv 1$ and $f_{\mathrm{M}\setminus C} \equiv 0$. Then, on $\tilde{U}$, we define

$$\varphi_i(q) = f(q)\tilde{\varphi}_i(q), \quad i = 1, \ldots, n,$$

and extend it by the 0-vector field to whole of M, i.e. $\varphi_i(q) = 0 \in \mathrm{E}_q$ for $q \in \mathrm{M} \setminus \tilde{U}$. □

Example 3.16.
A rank 1 vector bundle E (a *line bundle*) is trivial $\Leftrightarrow \exists$ nowhere vanishing $\varphi \in \Gamma(\mathrm{E})$

Example 3.17. Let $\mathrm{M} \subset \mathbb{R}^\ell$ submanifold of dimension n. Then a rank $\ell - n$ vector bundle NM (the *normal bundle of* M) is given by

$$\mathrm{N}_p\mathrm{M} = (\mathrm{NM}) = (\mathrm{T}_p\mathrm{M})^\perp \subset \mathrm{T}_p\mathbb{R}^\ell = \{p\} \times \mathbb{R}^\ell.$$

Fact: The normal bundle of a Moebius band is not trivial.

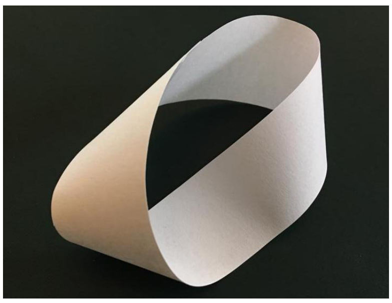


Figure 3.1: A Moebius band made from a piece of paper.

Example 3.18. The tangent bundle of $\mathbb{S}^2$ is not trivial - a fact known as the *hairy ball theorem*: Every vector field $X \in \Gamma(\mathrm{T}\,\mathbb{S}^2)$ has zeros.

Exercise 3.19.
Show that the tangent bundle $\mathrm{T}\mathbb{S}^3$ of the round sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ is trivial.
Hint: Show that the vector fields $\varphi_1(x_1, x_2, x_3, x_4) = (-x_2, x_1, x_4, -x_3)$, $\varphi_2(x_1, x_2, x_3, x_4) = (x_3, x_4, -x_1, -x_2)$ and $\varphi_3(x_1, x_2, x_3, x_4) = (-x_4, x_3, -x_2, x_1)$ form a frame of $\mathrm{T}\mathbb{S}^3$.

3.1 Vector fields as operators on functions

Let $X \in \Gamma(\mathrm{TM})$ and $f \in \mathscr{C}^\infty(\mathrm{M})$, then $Xf \colon \mathrm{M} \to \mathbb{R}$, $p \mapsto X_p f$, is smooth. So X can be viewed as a linear map

$$X : \mathscr{C}^\infty(\mathrm{M}) \to \mathscr{C}^\infty(\mathrm{M}),\ f \mapsto Xf.$$

Theorem 3.20 (Leibniz's rule)**.**
Let $f, g \in \mathscr{C}^\infty(\mathrm{M})$, $X \in \Gamma(\mathrm{TM})$, then

$$X(fg) = (Xf)g + f(Xg).$$

Definition 3.21 (Lie algebra)**.** *A Lie algebra is a vector space $\mathfrak{g}$ together with a skew bilinear map*

$$[.,.] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which satisfies the Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Theorem 3.22 (Lie algebra of endomorphisms)**.** *Let* V *be a vector space.* End(V) *together with the commutator*

$$[.,.] \colon \mathrm{End}(\mathrm{V}) \times \mathrm{End}(\mathrm{V}) \to \mathrm{End}(\mathrm{V}),\ [A, B] := AB - BA$$

forms a Lie algebra.

Proof. Certainly the commutaor is a skew bilinear map. Further,

$$\begin{aligned}[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = {} & A(BC - CB) - (BC - CB)A + B(CA - AC) \\ & - (CA - AC)B + C(AB - BA) - (AB - BA)C,\end{aligned}$$

which is zero since each term appears twice but with opposite sign. □

Theorem 3.23. *For all $f, g \in \mathscr{C}^\infty(\mathrm{M})$, $X, Y \in \Gamma(\mathrm{M})$, the following equality holds*

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

Lemma 3.24 (Schwarz lemma)**.** *Let $\varphi = (x_1, \ldots, x_n)$ be a coordinate chart, then*

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

Exercise 3.25.
Prove Schwarz lemma above.

is again a vector field !?

Thus, if $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x_j}$, we get

$$[X,Y] = \sum_{i,j} [a_i \frac{\partial}{\partial x_i}, b_j \frac{\partial}{\partial x_j}] = \sum_{i,j} (a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i}) = \sum_{i,j} (a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}) \frac{\partial}{\partial x_i}.$$

Thus $[X,Y] \in \Gamma(\mathrm{TM})$. In particular, we get the following theorem.

Theorem 3.26. *The set of sections on the tangent bundle* $\Gamma(\mathrm{TM}) \subset \mathrm{End}(\mathscr{C}^\infty(\mathrm{M}))$ *is a Lie subalgebra.*

Exercise 3.27.
Calculate the commutator $[X,Y]$ of the following vector fields on $\mathbb{R}^2 \setminus \{0\}$:

$$X = \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y}, \qquad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Write X and Y in polar coordinates $(r\cos\varphi, r\sin\varphi) \mapsto (r,\varphi)$.

Definition 3.28 (Push forward)**.**
Let $f\colon \mathrm{M} \to \mathrm{N}$ *be a diffeomorphism and* $X \in \Gamma(\mathrm{TM})$. *The push forward* $f_*X \in \Gamma(\mathrm{TN})$ *of* X *is defined by*

$$f_*X := df \circ X \circ f^{-1}$$

HW

Exercise 3.29.
Let $f\colon \mathrm{M} \to \mathrm{N}$ be a diffeomorphism, $X, Y \in \Gamma(\mathrm{TM})$. Show: $f_*[X,Y] = [f_*X, f_*Y]$.

3.2 Connections on vector bundles

Up to now we basically did *Differential Topology*. Now *Differential Geometry* begins, i.e. we study manifolds with additional ("geometric") structure.

Definition 3.30 (Connection)**.** *A connection on a vector bundle* $\mathrm{E} \to \mathrm{M}$ *is a bilinear map*

$$\nabla\colon \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{E}) \to \Gamma(\mathrm{E})$$

such that for all $f \in \mathscr{C}^\infty(\mathrm{M})$, $X \in \Gamma(\mathrm{TM})$, $\psi \in \Gamma(\mathrm{E})$,

(i) $\nabla_{fX}\psi = f\nabla_X\psi$

(ii) $\nabla_X f\psi = (Xf)\psi + f\nabla_X\psi$.

The proof of the following theorem will be postponed until we have established the existence of a so called *partition of unity*.

Theorem 3.31. *On every vector bundle* E *there is a connection* ∇.

Definition 3.32 (Parallel section)**.**
Let E $\to$ M *be a vector bundle with connection* ∇. *Then* $\psi \in \Gamma(\mathrm{E})$ *is called parallel if, for all* $X \in \mathrm{TM}$,

$$\nabla_X \psi = 0$$

Interlude:

Let ∇, $\tilde{\nabla}$ be two connections on E.
Define

$$A\colon \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{E}) \to \Gamma(\mathrm{E}) \text{ by } A_X\psi = \tilde{\nabla}_X\psi - \nabla_X\psi$$

Then A satisfies

$$A_{fX}\psi = \tilde{\nabla}_{fX}\psi - \nabla_{fX}\psi = fA_X\psi$$

and

$$A_X(f\psi) = \cdots = fA_X\psi.$$

Suppose we have $\omega \in \Gamma\mathrm{Hom}(\mathrm{TM}, \mathrm{End}\,\mathrm{E})$. Then define $B\colon \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{E}) \to \Gamma(\mathrm{E})$ by

$$(B_X\psi)_p = \omega_p(X_p)(\psi_p) \in \mathrm{E}_p.$$

Then

$$B_{fX}\psi = fB_X\psi, \qquad B_X(f\psi) = fB_X\psi.$$

Theorem 3.33 (Characterization of tensors)**.** *Let* E, F *be vector bundles over* M *and* $A\colon \Gamma(\mathrm{E}) \to \Gamma(\mathrm{F})$ *linear such that for all* $f \in \mathscr{C}^\infty(\mathrm{M})$, $\psi \in \Gamma(\mathrm{E})$ *we have*

$$A(f\psi) = fA(\psi).$$

Then there is $\omega \in \Gamma\mathrm{Hom}(\mathrm{E}, \mathrm{F})$ *such that* $(A\psi)_p = \omega_p(\psi_p)$ *for all* $\psi \in \Gamma(\mathrm{E})$, $p \in \mathrm{M}$.

Proof. Let $p \in \mathrm{M}$, $\tilde{\psi} \in \mathrm{E}_p$. We want to define ω by saying: Choose $\psi \in \Gamma(\mathrm{E})$ such that $\psi_p = \tilde{\psi}$. Then define $\omega_p(\tilde{\psi}) = (A\psi)_p$.

Claim: $(A\psi)_p$ depends only on ψ_p, i.e. if $\psi, \hat{\psi} \in \Gamma(\mathrm{E})$ with $\psi_p = \hat{\psi}_p$ then $(A\psi)_p = (A\hat{\psi})_p$, or in other words: $\psi \in \Gamma(\mathrm{E})$ with $\psi_p = 0$ then $(A\psi)_p = 0$.

Proof. choose a frame field $(\psi_1, \ldots, \psi_k)$ on some neighborhood and a function $f \in \mathscr{C}^\infty(\mathrm{M})$ such that $f\psi_1, \ldots, f\psi_k$ are globally defined sections and $f \equiv 1$ near p. Let $\psi \in \Gamma(\mathrm{E})$ with $\psi_p = 0$. This leads to

$$\psi|_U = a_1\psi_1 + \cdots + a_k\psi_k$$

with $a_1, \ldots, a_k \in \mathscr{C}^\infty(U)$. Then

$$\begin{aligned} f^2 A\psi &= A(f^2\psi) \\ &= A((fa_1)(f\psi_1) + \cdots + (fa_k)(f\psi_k)) \\ &= (fa_1)A(f\psi_1) + \cdots + (fa_k)A(f\psi_k)). \end{aligned}$$

Evaluation at p yields then $(A\psi)_p = 0$. □

□

Remark 3.34. In the following we keep this identification be tensors and tensorial maps in mind and just speak of tensors.

Thus the considerations above can be summarized by the following theorem.

Theorem 3.35. *Any two connections* ∇ *and* $\tilde{\nabla}$ *on a vector bundle* E *over* M *differ by a section of* $\mathrm{Hom}(\mathrm{TM}, \mathrm{End}\,\mathrm{E})$*:*

$$\tilde{\nabla} - \nabla \in \Gamma\mathrm{Hom}(\mathrm{TM}, \mathrm{End}\,\mathrm{E}).$$

Exercise 3.36 (Induced connections)**.**
Let E_i and F denote vector bundles with connections ∇^i and ∇, respectively. Show that the equation

$$(\hat{\nabla}_X T)(Y_1, \dots, Y_r) = \nabla_X(T(Y_1, \dots, Y_r)) - \sum_i T(Y_1, \dots, \nabla^i_X Y_i, \dots, Y_r)$$

for $T \in \Gamma(\mathrm{E}_1^* \otimes \cdots \otimes \mathrm{E}_r^* \otimes \mathrm{F})$ and vector fields $Y_i \in \Gamma(\mathrm{E}_i)$ defines a connection $\hat{\nabla}$ on the bundle of multilinear forms $\mathrm{E}_1^* \otimes \cdots \otimes \mathrm{E}_r^* \otimes \mathrm{F}$.

Remark 3.37. Note that, since an isomorphism $\rho\colon \mathrm{E} \to \tilde{\mathrm{E}}$ between vector bundles over M maps for each $p \in \mathrm{M}$ the fiber of E_p linearly to the fiber $\tilde{\mathrm{E}}_p$, the map ρ can be regarded as a section $\rho \in \Gamma\mathrm{Hom}(\mathrm{E}, \tilde{\mathrm{E}})$. If moreover E is equipped with a connection ∇ and $\tilde{\mathrm{E}}$ is equipped with a connection $\tilde{\nabla}$ we can speak then of parallel isomorphisms: ρ is called parallel if $\hat{\nabla}\rho = 0$, where $\hat{\nabla}$ is the connection on $\mathrm{Hom}(\mathrm{E}, \tilde{\mathrm{E}})$ induced by ∇ and $\tilde{\nabla}$ (compare Example 3.36 above).

Definition 3.38 (Metric)**.** *Let* $\mathrm{E} \to M$ *be a vector bundle and* $\mathrm{Sym}(\mathrm{E})$ *be the bundle whose fiber at* $p \in \mathrm{M}$ *consists of all symmetric bilinear forms* $\mathrm{E}_p \times \mathrm{E}_p \to \mathbb{R}$*. A metric on* E *is a section* $\langle .,. \rangle$ *of* $\mathrm{Sym}(\mathrm{E})$ *such that* $\langle .,. \rangle_p$ *is a Euclidean inner product for all* $p \in \mathrm{M}$*.*

Definition 3.39 (Euclidean vector bundle)**.** *A vector bundle together with a metric* $(\mathrm{E}, \langle .,. \rangle)$ *is called Euclidean vector bundle.*

Definition 3.40 (Metric connection)**.** *Let* $(\mathrm{E}, \langle .,. \rangle)$ *be a Euclidean vector bundle over* M*. Then a connection* ∇ *is called metric if for all* $\psi, \varphi \in \Gamma(\mathrm{E})$ *and* $X \in \Gamma(\mathrm{TM})$ *we have*

$$X\langle \psi, \varphi \rangle = \langle \nabla_X \psi, \varphi \rangle + \langle \psi, \nabla_X \varphi \rangle.$$

Exercise 3.41.
Let ∇ be a connection on a direct sum $\mathrm{E} = \mathrm{E}_1 \oplus \mathrm{E}_2$ of two vector bundles over M. Show that

$$\nabla = \begin{pmatrix} \nabla^1 & A \\ \tilde{A} & \nabla^2 \end{pmatrix},$$

where $\tilde{A} \in \Omega^1(\mathrm{M}, \mathrm{Hom}(\mathrm{E}_1, \mathrm{E}_2))$, $A \in \Omega^1(\mathrm{M}, \mathrm{Hom}(\mathrm{E}_2, \mathrm{E}_1))$ and ∇^i are connections on the bundles E_i.

Recall:

A rank k vector bundle $\mathrm{E} \to \mathrm{M}$ is called trivial if it is isomorphic to *the* trivial bundle $\mathrm{M} \times \mathbb{R}^k$. We know that

$$\mathrm{E}\ \textit{trivial} \quad \Leftrightarrow \quad \exists \varphi_1, \ldots, \varphi_k \in \Gamma(\mathrm{E})\colon \varphi_1(p), \ldots, \varphi_k(p)\ \textit{linearly independent for all}\ p \in \mathrm{M}.$$

The trivial bundle comes with a *trivial connection*

$$\nabla^{trivial} : \Gamma(\mathrm{M} \times \mathbb{R}^k) \ni \psi \quad \leftrightarrow \quad f = \pi_2 \circ \psi \in \mathscr{C}^\infty(\mathrm{M}, \mathbb{R}^k)$$

then

$$\nabla^{trivial}_X \psi \quad \leftrightarrow \quad d_X f = Xf$$

for $X \in \Gamma(\mathrm{TM})$. More precisely,

$$\nabla^{trivial}_X \psi = (\pi(X), Xf).$$

This clarified in the following the trivial connection often will be denoted just by d.

Every vector bundle E is *locally trivial*, i.e. each point $p \in \mathrm{M}$ has an open neighborhood U such that $\mathrm{E}|_U$ is trivial.

Definition 3.42 (Isomorphism of vector bundles with connection)**.** *An isomorphism between vector bundles with connection* (E, ∇) *and* $(\tilde{\mathrm{E}}, \tilde{\nabla})$ *is a vector bundle isomorphism* $\rho\colon \mathrm{E} \to \tilde{\mathrm{E}}$*, which is parallel, i.e. for all* $X \in \Gamma(\mathrm{TM})$*,* $\psi \in \Gamma(\mathrm{E})$*,*

$$\tilde{\nabla}_X(\rho \circ \psi) = \rho \circ (\nabla_X \psi).$$

Two vector bundles with connection are called isomorphic if there exists an isomorphism between them. A vector bundle with connection (E, ∇) *over* M *is called trivial if it is isomorphic to the trivial bundle* $(\mathrm{M} \times \mathbb{R}^k, d)$.

Remark 3.43. Note that $\psi \in \Gamma(\mathrm{M} \times \mathbb{R}^k)$ is parallel if $\pi \circ \psi$ is locally constant.

Theorem 3.44. *A vector bundle* E *with connection is trivial if and only if there exists a parallel frame field.*

Proof.

"⇒": Let $\rho\colon \mathrm{M} \times \mathbb{R}^k \to \mathrm{E}$ be a bundle isomorhism such that $\rho \circ d = \nabla \circ \rho$. Then

$$\phi_{ip} := \rho(p, e_i), \qquad i = 1, \ldots, k,$$

form a parallel frame.

"⇐": If we have a parallel frame field $\varphi_i \in \Gamma(\mathrm{E})$, then define

$$\rho\colon \mathrm{M} \times \mathbb{R}^k \to \mathrm{E}, \qquad \rho(p, v) := \sum v_i \varphi_i(p).$$

It is easily checked that ρ is the desired isomorphism.

□

Definition 3.45 (Flat vector bundle)**.** *A vector bundle* E *with connection is called flat if it is locally trivial as a vector bundle with connection, i.e. each point* $p \in \mathrm{M}$ *has an open neighborhood* U *such that* $\mathrm{E}|_U$ *(endowed with the connection inherited from* E*) is trivial.*
In other words: If there is a parallel frame field over U*.*

4. Differential Forms

4.1 Bundle-Valued Differential Forms

Definition 4.1 (Bundle-valued differential forms). *Let* $\mathrm{E} \to M$ *be a vector bundle. Then for* $\ell > 0$ *an E-valued* ℓ*-form* ω *is a section of the bundle* $\Lambda^\ell(\mathrm{M}, \mathrm{E})$ *whose fiber at* $p \in \mathrm{M}$ *is the vector space of multilinear maps* $\mathrm{T}_p\mathrm{M} \times \cdots \times \mathrm{T}_p\mathrm{M} \to \mathrm{E}_p$*, which are alternating, i.e. for* $i \neq j$

$$\omega_p(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_\ell) = -\omega_p(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_\ell).$$

Further, define $\Lambda^0(\mathrm{M}, \mathrm{E}) := \mathrm{E}$*. Consequently,* $\Omega^0(\mathrm{M}, \mathrm{E}) := \Gamma(\mathrm{E})$*.*

Remark 4.2. Each $\omega \in \Omega^\ell(\mathrm{M}, \mathrm{E})$ defines a tensorial map $\Gamma(\mathrm{TM})^\ell \to \Gamma(\mathrm{E})$ and vice versa.

Definition 4.3 (Exterior derivative). *Let* $\mathrm{E} \to \mathrm{M}$ *be a vector bundle with connection* ∇*. For* $\ell \geq 0$*, define the exterior derivative*

$$d^\nabla : \Omega^\ell(\mathrm{M}, \mathrm{E}) \to \Omega^{\ell+1}(\mathrm{M}, \mathrm{E})$$

for vectors $X_0, \ldots, X_\ell \in \Gamma(\mathrm{TM})$ *as follows:*

$$\begin{aligned} d^\nabla \omega(X_0, \ldots, X_\ell) := &\sum_i (-1)^i \nabla_{X_i}(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_\ell)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_\ell) \end{aligned}$$

Proof. Actually there are two things to be verified: $d^\nabla \omega$ is tensorial and alternating. First let us check it is tensorial:

$$\begin{aligned} d^\nabla \omega(X_0, \ldots, fX_k, \ldots, X_\ell) = &\sum_{i<k} (-1)^i \nabla_{X_i} \omega(X_0, \ldots, \hat{X}_i, \ldots, fX_k, \ldots, X_\ell) \\ &+ \nabla_{fX_k} \omega(X_0, \ldots, \hat{X}_k, \ldots, X_\ell) \\ &+ \sum_{i>k} (-1)^i \nabla_{X_i} \omega(X_0, \ldots, fX_k, \ldots, \hat{X}_i, \ldots, X_\ell) \\ &+ \sum_{i<j, i\neq k, j\neq k} (-1)^{i+j} \omega([X_i, X_j], \ldots, \hat{X}_i, \ldots, fX_k, \ldots, \hat{X}_j, \ldots, X_\ell) \\ &+ \sum_{i<k} (-1)^{i+k} \omega([X_i, fX_k], \ldots, \hat{X}_i, \ldots, \hat{X}_k, \ldots, X_\ell) \\ &+ \sum_{k<i} (-1)^{k+i} \omega([fX_k, fX_i], \ldots, \hat{X}_k, \ldots, \hat{X}_i, \ldots, X_\ell) \end{aligned}$$

$$
\begin{aligned}
&= f d^\nabla \omega(X_0, \dots, fX_k, \dots, X_\ell) \\
&\quad + \sum_{i \neq k} (-1)^i (X_i f) \omega(X_0, \dots, \hat{X}_i, \dots, X_\ell) \\
&\quad + \sum_{i<k} (-1)^{i+k} \omega((X_i f)X_k, \dots, \hat{X}_i, \dots, \hat{X}_k, \dots, X_\ell) \\
&\quad - \sum_{k<i} (-1)^{k+i} \omega((X_i f)X_k, \dots, \hat{X}_k, \dots, \hat{X}_i, \dots, X_\ell) \\
&= f d^\nabla \omega(X_0, \dots, fX_k, \dots, X_\ell).
\end{aligned}
$$

Next we want to see that $d^\nabla \omega$ is alternating. Since $d^\nabla \omega$ is tensorial we can test this on commuting vector fields, i.e $[X_i, X_j] = 0$. With this we get for $k < m$ that

$$
\begin{aligned}
d^\nabla \omega(X_0, \dots, X_m, \dots, X_k, \dots, X_\ell) &= \sum_{i<k} (-1)^i \nabla_{X_i} \omega(X_0, \dots, \hat{X}_i, \dots, X_m, \dots, X_k, \dots, X_\ell) \\
&\quad + (-1)^k \nabla_{X_m} \omega(X_0, \dots, \hat{X}_m, \dots, X_k, \dots, X_\ell) \\
&\quad + \sum_{k<i<m} (-1)^i \nabla_{X_i} \omega(X_0, \dots, X_m, \dots, \hat{X}_i, \dots, X_k, \dots, X_\ell) \\
&\quad + (-1)^m \nabla_{X_k} \omega(X_0, \dots, X_m, \dots, \hat{X}_m, \dots, X_\ell) \\
&\quad + \sum_{i>k} (-1)^i \nabla_{X_i} \omega(X_0, \dots, X_m, \dots, X_k, \dots, \hat{X}_i, \dots, X_\ell) \\
&= -\sum_{i<k} (-1)^i \nabla_{X_i} \omega(X_0, \dots, \hat{X}_i, \dots, X_k, \dots, X_m, \dots, X_\ell) \\
&\quad + (-1)^{k+(m-k-1)} \nabla_{X_m} \omega(X_0, \dots, X_k, \dots, \hat{X}_k, \dots, X_\ell) \\
&\quad - \sum_{k<i<m} (-1)^i \nabla_{X_i} \omega(X_0, \dots, X_k, \dots, \hat{X}_i, \dots, X_m, \dots, X_\ell) \\
&\quad + (-1)^{m+(m-k-1)} \nabla_{X_k} \omega(X_0, \dots, \hat{X}_k, \dots, X_m, \dots, X_\ell) \\
&\quad - \sum_{i>k} (-1)^i \nabla_{X_i} \omega(X_0, \dots, X_k, \dots, X_m, \dots, \hat{X}_i, \dots, X_\ell) \\
&= \sum_{i<k} (-1)^i \nabla_{X_i} \omega(X_0, \dots, \hat{X}_i, \dots, X_\ell) \\
&= -d^\nabla \omega(X_0, \dots, X_k, \dots, X_m, \dots, X_\ell),
\end{aligned}
$$

where the second equation follows by successively shifting the vector fields X_m resp. X_k to the right resp. left. □

1**-forms:**

Let $E \to M$ be a vector bundle with connection ∇, then $\Lambda^1(M, E) = \mathrm{Hom}(TM, E)$. We have $\Omega^0(M, E) = \Gamma(E)$. We obtain a 1-form by applying d^∇:

$$\Omega^0(M, E) \ni \psi \mapsto d^\nabla \psi = \nabla \psi \in \Omega^1(M, E).$$

As a special case we have $E = M \times \mathbb{R}$.
Then

$$\Gamma(M \times \mathbb{R}) \leftrightarrow \mathscr{C}^\infty(M)$$

and

$$\Lambda^1(M, M \times \mathbb{R}) = \mathrm{Hom}(TM, M \times \mathbb{R}) \leftrightarrow \mathrm{Hom}(TM, \mathbb{R}) = T^*M$$

So in this case $\Omega^1(M, M \times \mathbb{R}) \cong \Gamma(T^*M) = \Omega^1(M)$ (*ordinary* 1-forms are basically sections of T^*M). For $M = U \subset \mathbb{R}^n$ (open) we have the standard coordinates $x_i \colon U \to \mathbb{R}$ (projection to the i-component) $\rightsquigarrow dx_i \in \Omega^1(M)$.
Now let $X_i := \frac{\partial}{\partial x_i} \in \Gamma(TU)$ which as $\mathbb{R}^n$-valued functions is just the canonical basis $X_i = e_i$. Then $X_1, \dots, X_n$ is a frame and we have $dx_i(X_j) = \delta_{ij}$, thus $dx_1, \dots, dx_n$ is the frame of T^*U dual to $X_1, \dots, X_n$. So every 1-form is of the form:

$$\omega = a_1 dx_1 + \cdots + a_n dx_n, \quad a_1, \dots, a_n \in \mathscr{C}^\infty(U).$$

If $f \in \mathscr{C}^\infty(U)$, then $X_i f = \frac{\partial f}{\partial x_i}$. With a small computation we get

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

ℓ-forms:

Let $M \subset \mathbb{R}^n$ be open and consider again $E = M \times \mathbb{R}$. Then for $i_1, \dots, i_\ell$ define $dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Omega^\ell(M)$ by

$$dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}(X_1, \dots, X_\ell) := \det \begin{pmatrix} dx_{i_1}(X_1) & \cdots & dx_{i_1}(X_\ell) \\ \vdots & \ddots & \vdots \\ dx_{i_\ell}(X_1) & \cdots & dx_{i_\ell}(X_\ell) \end{pmatrix}.$$

Note: If $i_\alpha = i_\beta$ for $\alpha \neq \beta$, then $dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} = 0$. If $\sigma \colon \{1, \dots, \ell\} \to \{1, \dots, \ell\}$ is a permutation, we have

$$dx_{i_{\sigma_1}} \wedge \cdots \wedge dx_{i_{\sigma_\ell}} = \operatorname{sign} \sigma \; dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}.$$

Theorem 4.4. *Let $U \subset \mathbb{R}^n$ be open. The ℓ-forms $dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}$ for $1 \le i_1 < \cdots < i_\ell \le n$ are a frame field for $\Lambda^\ell(U)$, i.e. each $\omega \in \Omega^\ell(U)$ can be uniquely written as*

$$\omega = \sum_{1 \le i_1 < \cdots < i_\ell \le n} a_{i_1 \cdots i_\ell} \, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}$$

with $a_{i_1 \cdots i_\ell} \in \mathscr{C}^\infty(U)$. In fact,

$$a_{i_1 \cdots i_\ell} = \omega\Big(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_\ell}}\Big).$$

Proof. For uniqueness note that

$$dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}\Big(\frac{\partial}{\partial x_{j_1}}, \dots, \frac{\partial}{\partial x_{j_\ell}}\Big) = \det \begin{pmatrix} \delta_{i_1 j_1} & \cdots & \delta_{i_1 j_\ell} \\ \vdots & \ddots & \vdots \\ \delta_{i_\ell j_1} & \cdots & \delta_{i_\ell j_\ell} \end{pmatrix} = \begin{cases} 1 & \text{if } \{i_1, \dots, i_\ell\} = \{j_1, \dots, j_\ell\}, \\ 0 & \text{else.} \end{cases}$$

Existence we leave as an exercise. ◻

Theorem 4.5. *Let $U \subset \mathbb{R}^n$ be open and $\omega = \sum_{1 \le i_1 < \cdots < i_\ell \le n} a_{i_1 \cdots i_\ell} \, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Omega^\ell(U)$, then*

$$d\omega = \sum_{1 \le i_1 < \cdots < i_\ell \le n} \sum_{i=1}^{n} \frac{\partial a_{i_1 \cdots i_\ell}}{\partial x_i} \, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}.$$

Proof. By Theorem 4.4 it is enough to show that for all $1 \leq j_0 < \cdots < j_\ell \leq n$

$$\begin{aligned} d\omega(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_\ell}}) &= \sum_{1\leq i_1 < \cdots < i_\ell \leq n} \sum_{i=1}^{n} \frac{\partial a_{i_1\cdots i_\ell}}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_\ell}}) \\ &= \sum_{k=0}^{\ell} \frac{\partial a_{j_0\cdots \hat{j}_k \cdots j_\ell}}{\partial x_{j_k}} dx_{j_k} \wedge dx_{j_0} \cdots \wedge \widehat{dx_{j_k}} \cdots \wedge dx_{j_\ell}(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_\ell}}) \\ &= \sum_{k=0}^{\ell} (-1)^k \frac{\partial a_{j_0\cdots \hat{j}_k \cdots j_\ell}}{\partial x_{j_k}}. \end{aligned}$$

But we also get this sum if we apply the definition and use that $[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_m}] = 0$. □

Example 4.6. Let $M = U \subset \mathbb{R}^3$ be open. Then every $\sigma \in \Omega^2(M)$ can be uniquely written as

$$\sigma = a_1 dx_2 \wedge dx_3 + a_2 dx_3 \wedge dx_1 + a_3 dx_1 \wedge dx_2.$$

Let $\sigma = d\omega$ with $\omega = v_1 dx_1 + v_2 dx_2 + v_3 dx_3$. Then

$$d\omega(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_i}\omega(\frac{\partial}{\partial x_j}) - \frac{\partial}{\partial x_j}\omega(\frac{\partial}{\partial x_i}) = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}.$$

Thus we get that $a = \mathrm{curl}(v)$.

The proofs of Theorem 4.4 and Theorem 4.5 directly carry over to bundle-valued forms.

Theorem 4.7. *Let $U \subset \mathbb{R}^n$ be open and $E \to U$ be a vector bundle with connection ∇. Then $\omega \in \Omega^\ell(U, E)$ can be uniquely written as*

$$\omega = \sum_{1\leq i_1 < \cdots < i_\ell \leq n} \psi_{i_1\cdots i_\ell} dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}, \quad \psi_{i_1\cdots i_\ell} \in \Gamma(E).$$

Moreover,

$$d^\nabla \omega = \sum_{1\leq i_1 < \cdots < i_\ell \leq n} \sum_{i=1}^{n} (\nabla_{\frac{\partial}{\partial x_i}} \psi_{i_1\cdots i_\ell}) \, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}.$$

Exercise 4.8.
Let $M = \mathbb{R}^2$. Let $J \in \Gamma(\mathrm{End}TM)$ be the 90° rotation and $\det \in \Omega^2(M)$ denote the determinant. Define $*\colon \Omega^1(M) \to \Omega^1(M)$ by $*\omega(X) = -\omega(JX)$. Show that

a) for all $f \in \mathscr{C}^\infty(M)$, $d * df = (\Delta f) \det$, where $\Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f$,

b) $\omega \in \Omega^1(M)$ is *closed* (i.e. $d\omega = 0$), if and only if ω is *exact* (i.e. $\omega = df$).

4.2 Wedge product

Let U, V, W be vector bundles over M. Let $\omega \in \Omega^k(M, U)$, $\eta \in \Omega^\ell(M, V)$. We want to define

$$\omega \wedge \eta \in \Omega^{k+\ell}(M, W).$$

Therefore we need a multiplication $*\colon U_p \times V_p \to W_p$ bilinear such that for $\psi \in \Gamma(U), \phi \in \Gamma(V)$ such that $\psi * \phi\colon p \mapsto \psi_p *_p \phi_p$ is smooth, i.e. $\psi * \phi \in \Gamma(W)$. In short,

$$* \in \Gamma(U^* \otimes V^* \otimes W).$$

Example 4.9.

a) Most standard case: $U = M \times \mathbb{R} = V$, $*$ ordinary multiplication in $\mathbb{R}$.

b) Also useful: $U = M \times \mathbb{R}^{k\times \ell}$, $V = M \times \mathbb{R}^{\ell\times m}$, $W = M \times \mathbb{R}^{k\times m}$, $*$ matrix multiplication.

c) Another case: $U = \mathrm{End}(E)$, $V = W = E$, $*$ evaluation of endomorphisms on vectors, i.e. $(A * \psi)_p = A_p(\psi_p)$.

Definition 4.10 (Wedge product). *Let* U, V, W *be vector bundles over* M *and* $* \in \Gamma(U^* \otimes V^* \otimes W)$. *For two forms* $\omega \in \Omega^k(M,U)$ *and* $\eta \in \Omega^\ell(M,V)$ *the wedge product* $\omega \wedge \eta \in \Omega^{k+\ell}(M,W)$ *is then defined as follows*

$$\omega \wedge \eta(X_1,\dots,X_{k+\ell}) := \frac{1}{k!\ell!} \sum_{\sigma\in S_{k+\ell}} \operatorname{sgn}\sigma\, \omega(X_{\sigma_1},\dots,X_{\sigma_k}) * \eta(X_{\sigma_{k+1}},\dots,X_{\sigma_{k+\ell}}).$$

Example 4.11 (Wedge product of 1-forms)**.** For $\omega, \eta \in \Omega^1(M)$ we have

$$\omega \wedge \eta(X,Y) = \omega(X)\eta(Y) - \omega(Y)\eta(X).$$

Theorem 4.12. *Let* U, V, W *be vector bundles over* M, $* \in \Gamma(U^* \otimes V^* \otimes W)$, $\tilde{*} \in \Gamma(V^* \otimes U^* \otimes W)$ *such that* $\psi * \phi = \phi \,\tilde{*}\, \psi$ *for all* $\psi \in \Gamma(U)$ *and* $\phi \in \Gamma(V)$, *then for* $\omega \in \Omega^k(M,U)$, $\eta \in \Omega^\ell(M,V)$ *we have*

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

Proof. The permutation $\rho\colon \{1,\dots,k+\ell\} \to \{1,\dots,k+\ell\}$ with $(1,\dots,k,k+1,\dots,k+\ell) \mapsto (k+1,\dots,k+\ell,1,\dots,k)$ needs $k\ell$ transpositions, i.e. $\operatorname{sgn}\rho = (-1)^{k\ell}$. Thus

$$\begin{aligned}
\omega \wedge \eta(X_1,\dots,X_{k+\ell}) &= \frac{1}{k!\ell!} \sum_{\sigma\in S_{k+\ell}} \operatorname{sgn}\sigma\, \omega(X_{\sigma_1},\dots,X_{\sigma_k}) * \eta(X_{\sigma_{k+1}},\dots,X_{\sigma_{k+\ell}}) \\
&= \frac{1}{k!\ell!} \sum_{\sigma\in S_{k+\ell}} \operatorname{sgn}\sigma\, \eta(X_{\sigma_{k+1}},\dots,X_{\sigma_{k+\ell}}) \,\tilde{*}\, \omega(X_{\sigma_1},\dots,X_{\sigma_k}) \\
&= \frac{1}{k!\ell!} \sum_{\sigma\in S_{k+\ell}} \operatorname{sgn}(\sigma\circ\rho)\, \eta(X_{\sigma_{\rho_{k+1}}},\dots,X_{\sigma_{\rho_{k+\ell}}}) \,\tilde{*}\, \omega(X_{\sigma_{\rho_1}},\dots,X_{\sigma_{\rho_k}}) \\
&= \frac{(-1)^{k\ell}}{k!\ell!} \sum_{\sigma\in S_{k+\ell}} \operatorname{sgn}\sigma\, \eta(X_{\sigma_1},\dots,X_{\sigma_k}) \,\tilde{*}\, \omega(X_{\sigma_{k+1}},\dots,X_{\sigma_{k+\ell}}) \\
&= (-1)^{k\ell}\, \eta \wedge \omega(X_1,\dots,X_{k+\ell}).
\end{aligned}$$

□

Remark 4.13. In particular the above theorem holds for symmetric tensors $* \in \Gamma(U^* \otimes U^* \otimes V)$.

Theorem 4.14. *Let* $E_1, \dots, E_6$ *be vector bundles over* M. *Suppose that* $* \in \Gamma(E_1^* \otimes E_2^* \otimes E_4^*)$, $\tilde{*} \in \Gamma(E_4^* \otimes E_3^* \otimes E_5)$, $\mathring{*} \in \Gamma(E_1 \otimes E_6 \otimes E_5)$ *and* $\hat{*} \in \Gamma(E_2^* \otimes E_3^* \otimes E_6)$ *be associative, i.e.*

$$(\psi_1 * \psi_2) \,\tilde{*}\, \psi_3 = \psi_1 \,\mathring{*}\, (\psi_2 \,\hat{*}\, \psi_3), \text{ for all } \psi_1 \in \Gamma(E_1), \psi_1 \in \Gamma(E_2), \psi_1 \in \Gamma(E_3).$$

Then for $\omega_1 \in \Omega^{k_1}(M, E_1)$, $\omega_2 \in \Omega^{k_2}(M, E_2)$ *and* $\omega_3 \in \Omega^{k_3}(M, E_3)$ *we have*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

Proof. To simplify notation: $E_1 = \cdots = E_6 = M \times \mathbb{R}$ with ordinary multiplication of real numbers. $\omega_1 = \alpha$, $\omega_2 = \beta$, $\omega_3 = \gamma$, $k_1 = k$, $k_2 = \ell$, $k_3 = m$.

$$\alpha \wedge (\beta \wedge \gamma)(X_1, \dots, X_{k+\ell+m}) = \frac{1}{k!(\ell+m)!} \sum_{\sigma \in S_{k+\ell+m}} \operatorname{sgn} \sigma \, \alpha(X_{\sigma_1}, \dots, X_{\sigma_k})$$
$$\cdot \frac{1}{m!} \sum_{\rho \in S_{\ell+m}} \operatorname{sgn} \rho \, \beta(X_{\sigma_{k+\rho_1}}, \dots, X_{\sigma_{k+\rho_\ell}}) \gamma(X_{\sigma_{k+\rho_{\ell+1}}}, \dots, X_{\sigma_{k+\rho_{\ell+m}}})$$

Observe: Fix $\sigma_1, \dots, \sigma_k$. Then $\sigma_{k+1}, \dots, \sigma_{k+\ell+m}$ already account for all possible permutations of the remaining indices. In effect we get the same term $(\ell+m)!$ (number of elements in $S_{\ell+m}$) many times. So:

$$\alpha \wedge (\beta \wedge \gamma)(X_1, \dots, X_{k+\ell+m}) = \frac{1}{k!\ell!m!} \sum_{\sigma \in S_{k+\ell+m}} \operatorname{sgn} \sigma \, \alpha(X_{\sigma_1}, \dots, X_{\sigma_k})$$
$$\cdot \beta(X_{\sigma_{k+1}}, \dots, X_{\sigma_{k+\ell}}) \gamma(X_{\sigma_{k+\ell+1}}, \dots, X_{\sigma_{k+\ell+m}}).$$

Calculation of $(\alpha \wedge \beta) \wedge \gamma$ gives the same result. $\square$

Important special case:

On a chart neighborhood (U, φ) of M with $\varphi = (x_1, \dots, dx_n)$ we have

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(Y_1, \dots, Y_k) = \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \, dx_{i_1}(Y_{\sigma_1}) \cdots dx_{i_k}(Y_{\sigma_k}) = \det\Big(dx_{i_j}(Y_k)\Big)_{j,k},$$

as was defined previously. In particular, for a bundle-valued form $\omega \in \Omega^\ell(M, E)$ we obtain with Theorem 4.7 that

$$\omega|_U = \sum_{1 \le i_1 < \cdots < i_\ell \le n} \psi_{i_1 \cdots i_\ell} \, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}, \quad \psi_{i_1 \cdots i_\ell} \in \Gamma(E|_U),$$

and

$$(d^\nabla \omega)\Big|_U = \sum_{1 \le i_1 < \cdots < i_\ell \le n} d^\nabla \psi_{i_1 \cdots i_\ell} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}.$$

Theorem 4.15 (Product rule). *Let* E_1, E_2 *and* E_3 *be vector bundles over* M *with connections* ∇^1, ∇^2 *and* ∇^3, *respectively. Let* $* \in \Gamma(E_1^* \otimes E_2^* \otimes E_3)$ *be parallel, i.e.* $\nabla^3(\psi * \varphi) = (\nabla^1 \psi) * \varphi + \psi * (\nabla^2 \varphi)$ *for all* $\psi \in \Gamma(E_1)$ *and* $\varphi \in \Gamma(E_2)$.. *Then, if* $\omega \in \Omega^k(M, E_1)$ *and* $\eta \in \Omega^\ell(M, E_2)$, *we have*

$$d^{\nabla^3}(\omega \wedge \eta) = (d^{\nabla^1} \omega) \wedge \eta + (-1)^k \omega \wedge (d^{\nabla^2} \eta).$$

Proof. It is enough to show this locally. For $\omega = \psi\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\eta = \varphi\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}$,

$$\begin{aligned}
d^{\nabla^3}(\omega \wedge \eta) &= d^{\nabla^3}(\psi * \varphi\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}) \\
&= d^{\nabla^3}(\psi * \varphi) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= ((d^{\nabla^1}\psi) * \varphi + \psi * d^{\nabla^2}\varphi) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= (d^{\nabla^1}\psi) * \varphi \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&\quad + \psi * (d^{\nabla^2}\varphi) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= d^{\nabla^1}\psi \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge \varphi\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&\quad + (-1)^k \psi \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge d^{\nabla^2}\varphi \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= (d^{\nabla^1}\omega) \wedge \eta + (-1)^k \omega \wedge (d^{\nabla^2}\eta).
\end{aligned}$$

Since d^{∇} is $\mathbb{R}$-linear and the wedge product is bilinear the claim follows. □

5. Pullback

Motivation: A geodesic in M is a curve γ without acceleration, i.e. $\gamma'' = (\gamma')' = 0$.

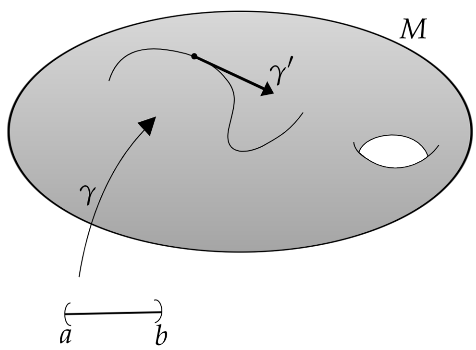
But what a map is γ'? What is the second prime?

We know that $\gamma'(t) \in T_{\gamma(t)}M$. Modify γ' slightly by

$$\widehat{\gamma'}(t) = (t, \gamma'(t)) \quad \text{meaning that} \quad \widehat{\gamma'} \in \Gamma(\gamma^* TM).$$

Right now $\gamma^* TM$ is just a vector bundle over $(-\varepsilon, \varepsilon)$. If we had a connection $\widehat{\nabla}$ then we can define

$$\gamma'' = \widehat{\nabla}_{\frac{\partial}{\partial t}} \widehat{\gamma'}.$$



Definition 5.1 (Pullback of forms). *Let $f : M \to \tilde{M}$ be smooth and $\omega \in \Omega^k(\tilde{M}, E)$. Then define*

$$f^*\omega \in \Omega^k(M, f^*E)$$

by

$$(f^*\omega)(X_1, \ldots, X_k) := (p, \omega(df(X_1), \ldots, df(X_k)))$$

for all $p \in M$, $X_1, \ldots, X_k \in T_pM$.
For $\psi \in \Omega^0(\tilde{M}, E)$ we have $f^*\psi = (\mathrm{Id}, \psi \circ f)$.

For ordinary k-forms $\omega \in \Omega^k(\tilde{M}) \cong \Omega^k(\tilde{M}, \tilde{M} \times \mathbb{R})$:

$$(f^*\omega)(X_1, \ldots, X_k) = \omega(df(X_1), \ldots, df(X_k)).$$

Let $E \to \tilde{M}$ be a vector bundle with connection $\tilde{\nabla}$, $f \colon M \to \tilde{M}$.

Theorem 5.2. *There is a unique connection*

$$\nabla =: f^*\tilde{\nabla}$$

on $f^*\mathrm{E}$ *such that for all* $\psi \in \Gamma(\mathrm{E})$, $X \in \mathrm{T}_p\mathrm{M}$ *we have* $\nabla_X(f^*\psi) = \big(p, \tilde{\nabla}_{df(X)}\psi\big)$.
In other words

$$\nabla(f^*\psi) = (f^*\tilde{\nabla})(f^*\psi) = f^*(\tilde{\nabla}\psi).$$

Proof. For uniqueness we choose a local frame field $\varphi_1, \dots, \varphi_k$ around $f(p)$ defined on $V \subset \mathrm{N}$ and an open neighborhood $U \subset \mathrm{M}$ of p such that $f(U) \subset V$.
Then for any $\psi \in \Gamma\big((f^*\mathrm{E})|_U\big)$ there are $g_1, \dots, g_k \in \mathscr{C}^\infty(U)$ such that $\psi = \sum_j g_j f^*\varphi_j$. If a connection ∇ on $f^*\mathrm{E}$ has the desired property then, for $X \in \mathrm{T}_p\mathrm{M}$,

$$\begin{aligned}\nabla_X\psi &= \sum_j \big((Xg_j)f^*\varphi_j + g_j\nabla_X(f^*\varphi_j)\big)\\ &= \sum_j \big((Xg_j)f^*\varphi_j + g_j(p, \tilde{\nabla}_{df(X)}\varphi_j)\big)\\ &= \sum_j \big((Xg_j)f^*\varphi_j + g_j\sum_k (p, \omega_{jk}(X)\varphi_k)\big)\\ &= (p, \sum_j \big((Xg_j)\varphi_j \circ f + g_j\sum_k \omega_{jk}(X)\varphi_k \circ f\big),\end{aligned}$$

where

$$\tilde{\nabla}_{df(X)}\varphi_j = \sum_k \omega_{jk}(X)\varphi_k \circ f,$$

with $\omega_{jk} \in \Omega^1(U)$. For existence check that this formula *defines* a connection. □

Theorem 5.3. *Let* $\omega \in \Omega^k(\mathrm{M}, \mathrm{U})$, $\eta \in \Omega^\ell(\mathrm{M}, \mathrm{V})$ *and* $* \in \Gamma(\mathrm{U}^* \otimes \mathrm{V}^* \otimes \mathrm{W})$, *then*

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta.$$

Proof. Trivial. □

Theorem 5.4. *Let* E *be a vector bundle with connection* ∇ *over* $\tilde{\mathrm{M}}$, $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$, $\omega \in \Omega^k(\tilde{\mathrm{M}}, \mathrm{E})$, *then*

$$d^{f^*\nabla}(f^*\omega) = f^*(d^\nabla\omega).$$

Proof. Without loss of generality we can assume that $\tilde{\mathrm{M}} \subset \mathbb{R}^n$ is open and that ω is of the form

$$\omega = \sum_{1 \le i_1 < \dots < i_k \le n} \psi_{i_1 \cdots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Then

$$\begin{aligned}f^*\omega &= \sum_{1 \le i_1 < \dots < i_k \le n} (f^*\psi_{i_1 \cdots i_k}) f^*dx_{i_1} \wedge \dots \wedge f^*dx_{i_k},\\ d^\nabla\omega &= \sum_{1 \le i_1 < \dots < i_k \le n} \nabla\psi_{i_1 \cdots i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.\end{aligned}$$

Hence

$$\begin{aligned} f^*d^{\nabla}\omega &= \sum_{1\leq i_1<\cdots<i_k\leq n} f^*(\nabla\psi_{i_1\cdots i_k})\wedge f^*dx_{i_1}\wedge\cdots\wedge f^*dx_{i_k} \\ &= \sum_{1\leq i_1<\cdots<i_k\leq n} (f^*\nabla f^*\psi_{i_1\cdots i_k})\wedge dx_{i_1}\circ df\wedge\cdots\wedge dx_{i_k}\circ df \\ &= \sum_{1\leq i_1<\cdots<i_k\leq n} (d^{f^*\nabla} f^*\psi_{i_1\cdots i_k})\wedge d(x_{i_1}\circ f)\wedge\cdots\wedge d(x_{i_k}\circ f) \\ &= d^{f^*\nabla}(f^*\omega). \end{aligned}$$

□

Exercise 5.5.
Consider the polar coordinate map $f\colon \{(r,\theta)\in\mathbb{R}^2 \mid r>0\}\to\mathbb{R}^2$ given by $f(r,\theta) := (r\cos\theta, r\sin\theta) = (x,y)$. Show that

$$f^*(x\,dx + y\,dy) = r\,dr \qquad \text{and} \qquad f^*(x\,dy - y\,dx) = r^2\,d\theta.$$

Theorem 5.6 (Pullback metric)**.** *Let* $\mathrm{E}\to\tilde{\mathrm{M}}$ *be a Euclidean vector bundle with bundle metric* g *and* $f\colon \mathrm{M}\to\tilde{\mathrm{M}}$*. Then on* $f^*\mathrm{E}$ *there is a unique metric* f^*g *such that*

$$(f^*g)(f^*\psi, f^*\phi) = f^*g(\psi,\phi)$$

and f^*g *is parallel with respect to the pullback connection* $f^*\nabla$*.*

Exercise 5.7.
Prove Theorem 5.6.

6. Curvature

Consider the trivial bundle $E = M \times \mathbb{R}^k$, then

$$f \in \mathscr{C}^\infty(M, \mathbb{R}^k) \quad \leftrightarrow \quad \psi \in \Gamma(E) \text{ by } f \leftrightarrow \psi = (\mathrm{Id}_M, f)$$

On E we have the trivial connection ∇:

$$\psi = (\mathrm{Id}_M, f) \in \Gamma(E), \quad X \in \Gamma(TM).$$

This leads to the definition

$$\nabla_X \psi := (\mathrm{Id}_M, Xf).$$

Claim: This ∇ satisfies for all $X, Y \in \Gamma(TM)$, $\psi \in \Gamma(E)$:

$$\nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi = \nabla_{[X,Y]} \psi.$$

Proof. The fact that $\nabla_Y \psi = (\mathrm{Id}_M, Yf)$ and $\nabla_X \nabla_Y \psi = (\mathrm{Id}_M, XYf)$ yields

$$\nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi = (\mathrm{Id}_M, [X,Y]f) = \nabla_{[X,Y]} \psi.$$

□

In the case that $M \subset \mathbb{R}^n$ open, $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j}$ this leads to $[X,Y] = 0$ and the above formula says

$$\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \psi = \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \psi.$$

The equation $\nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]} \psi = 0$ reflects the fact that for the trivial connection partial derivatives commute.

Define a map

$$\tilde{R}^\nabla \colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

by

$$(X, Y, \psi) \mapsto \tilde{R}^\nabla(X,Y)\psi := \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]} \psi.$$

Theorem 6.1. *Let* E *be a vector bundle with connection* ∇*. Then for all* $X, Y \in \Gamma(TM)$ *and* $\psi \in \Gamma(E)$ *we have*

$$\tilde{R}^\nabla(X,Y)\psi = d^\nabla d^\nabla \psi(X,Y).$$

Proof. In fact it is

$$\begin{aligned} d^\nabla(d^\nabla \psi)(X,Y) &= \nabla_X(d^\nabla \psi(Y)) - \nabla_Y(d^\nabla \psi(X)) - d^\nabla \psi([X,Y]) \\ &= \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]} \psi. \end{aligned}$$

□

Theorem 6.2 (Curvature tensor). *Let ∇ be a connection on a vector bundle* E *over* M*. The map $\tilde{R}^\nabla$ is tensorial in X, Y and ψ. The corresponding tensor $R^\nabla \in \Omega^2(\mathrm{M}, \mathrm{End}\mathrm{E})$ such that*

$$[\tilde{R}^\nabla(X,Y)\psi]_p = R^\nabla(X_p, Y_p)\psi_p$$

is called the curvature tensor of ∇.

Proof. Tensoriality in X and Y follows from the last theorem. Remains to show that $\tilde{R}^\nabla$ is tensorial in ψ:

$$\begin{aligned}
\tilde{R}^\nabla(X,Y)(f\psi) &= \nabla_X\nabla_Y(f\psi) - \nabla_Y\nabla_X(f\psi) - \nabla_{[X,Y]}(f\psi)\\
&= \nabla_X((Yf)\psi + f\nabla_Y\psi) - \nabla_Y((Xf)\psi + f\nabla_X\psi))\psi - (([X,Y]f)\psi + f\nabla_{[X,Y]}\psi)\\
&= X(Yf)\psi + (Yf)\nabla_X\psi + (Xf)\nabla_Y\psi + f\nabla_X\nabla_Y\psi - Y(Xf)\psi\\
&\quad - (Xf)\nabla_Y\psi - (Yf)\nabla_Y\psi \ - f\nabla_Y\nabla_X\psi - ([X,Y]f)\psi - f\nabla_{[X,Y]}\psi\\
&= f\tilde{R}^\nabla(X,Y)\psi.
\end{aligned}$$

□

Exercise 6.3.
Let E $\to$ M be a vector bundle with connection ∇, $\psi \in \Gamma(E)$ and $f\colon \mathrm{N} \to \mathrm{M}$. Then

$$(f^*R^\nabla)(f^*\tilde{\psi}) = f^*(R^\nabla\psi) = R^{f^*\nabla}f^*\psi.$$

Lemma 6.4. *Given $\hat{X}_1, \dots, \hat{X}_k \in \mathrm{T}_p\mathrm{M}$, then there are vector fields $X_1, \dots, X_k \in \Gamma(\mathrm{TM})$ such that*

$$X_{1p} = \hat{X}_1, \dots, X_{kp} = \hat{X}_k$$

and there is a neighborhood $U \ni p$ such that

$$[X_i, X_j]\big|_U = 0.$$

Proof. We have already seen that we can extend coordinate frames to the whole manifold. This yields n vector fields Y_i such that $[Y_i, Y_j]$ vanishes on a neighborhood of p. Since there Y_i form a frame. Then we can build linear combinations of Y_i (constant coefficients) to obtain the desired fields. □

Theorem 6.5.
Let E $\to$ M *be a vector bundle with connection ∇. For each $\omega \in \Omega^k(\mathrm{M}, \mathrm{E})$*

$$d^\nabla d^\nabla \omega = R^\nabla \wedge \omega.$$

Proof. Let $p \in \mathrm{M}$, $\hat{X}_1, \dots, \hat{X}_{k+2} \in \mathrm{T}_p\mathrm{M}$. Choose $X_1, \dots, X_{k+2} \in \Gamma(\mathrm{TM})$ such that $X_{ip} = \hat{X}_i$ and near p we have $[X_i, X_j] = 0$, $i, j \in \{1, \dots, k+2\}$. The left side is tensorial, so we can use $X_1, \dots, X_{k+2}$ to evaluate $d^\nabla d^\nabla\omega(\hat{X}_1, \dots, \hat{X}_{k+2})$. Then $i_j \in \{1, \dots, k+2\}$

$$d^\nabla\omega(X_{i_0}, \dots, X_{i_k}) = \sum_{j=0}^{k}(-1)^j \nabla_{X_{i_j}}\omega(X_{i_0}, \dots, \hat{X}_{i_j}, \dots, X_{i_k}).$$

Then

$$\begin{aligned} d^\nabla d^\nabla \omega(X_1,\dots,X_{k+2}) &= \sum_{i<j}(-1)^{i+j}\nabla_{X_i}\nabla_{X_j}\omega(X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+2}) \\ &\quad+\sum_{j<i}(-1)^{i+j+1}\nabla_{X_i}\nabla_{X_j}\omega(X_1,\dots,\hat{X}_j,\dots,\hat{X}_i,\dots,X_{k+2}) \\ &= \sum_{i<j}(-1)^{i+j}(\nabla_{X_i}\nabla_{X_j}-\nabla_{X_j}\nabla_{X_i})\omega(X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+2}) \\ &= \sum_{i<j}(-1)^{i+j}R^\nabla(X_i,X_j)\omega(X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+2}). \end{aligned}$$

On the other hand

$$R^\nabla\wedge\omega(X_1,\dots,X_{k+2}) = \frac{1}{2\cdot k!}\sum_{\sigma\in S_{k+2}}\operatorname{sgn}\sigma R^\nabla(X_{\sigma_1},X_{\sigma_2})\omega(X_{\sigma_3},\dots,X_{\sigma_{k+2}}).$$

For $i,j\in\{1,\dots,k+2\}$, $i\neq j$ define

$$A_{\{i,j\}} := \big\{\sigma\in S_{k+2}\mid\{\sigma_1,\sigma_2\}=\{i,j\}\big\}.$$

For $i<j$ define $\sigma^{ij}\in S_{k+2}$ by $\sigma_1^{ij}=i$ and $\sigma_2^{ij}=j$, $\sigma_3^{ij}<\dots<\sigma_{k+2}^{ij}$, i.e.

$$\sigma^{ij}=(i,j,3,\dots,\hat{i},\dots,\hat{j},\dots,k+2).$$

In particular we find that $\operatorname{sgn}\sigma^{ij}=(-1)^{i+j}$. Further

$$A_{\{i,j\}} = \underbrace{\{\sigma^{ij}\circ\rho\mid\rho\in S_{k+2},\,\rho_1=1,\,\rho_2=2\}}_{=:A^+_{\{i,j\}}}\cup\underbrace{\{\sigma^{ij}\circ\rho\mid\rho\in S_{k+2},\,\rho_1=2,\,\rho_2=1\}}_{=:A^-_{\{i,j\}}}.$$

Note, $\operatorname{sgn}(\sigma^{ij}\circ\rho)=(-1)^{i+j}\operatorname{sgn}\rho$. With this we get

$$\begin{aligned} R^\nabla\wedge\omega(X_1,\dots,X_{k+2}) &= \frac{1}{2\cdot k!}\sum_{i<j}\sum_{\sigma\in A_{\{i,j\}}}\operatorname{sgn}\sigma R^\nabla(X_{\sigma_1},X_{\sigma_2})\omega(X_{\sigma_3},\dots,X_{\sigma_{k+2}}) \\ &= \frac{1}{2\cdot k!}\sum_{i<j}\Bigg(\sum_{\sigma\in A^+_{\{i,j\}}}\operatorname{sgn}\sigma R^\nabla(X_{\sigma_1},X_{\sigma_2})\omega(X_{\sigma_3},\dots,X_{\sigma_{k+2}}) \\ &\qquad+\sum_{\sigma\in A^-_{\{i,j\}}}\operatorname{sgn}\sigma R^\nabla(X_{\sigma_1},X_{\sigma_2})\omega(X_{\sigma_3},\dots,X_{\sigma_{k+2}})\Bigg) \\ &= \frac{1}{2\cdot k!}\sum_{i<j}\Bigg(\sum_{\rho\in S_{k+2},\rho_1=1,\rho_2=2}(-1)^{i+j}\operatorname{sgn}\rho R^\nabla(X_i,X_j)\omega(X_{\sigma^{ij}_{\rho_3}},\dots,X_{\sigma^{ij}_{\rho_{k+2}}}) \\ &\qquad+\sum_{\rho\in S_{k+2},\rho_1=2,\rho_2=1}(-1)^{i+j}\operatorname{sgn}\rho R^\nabla(X_j,X_i)\omega(X_{\sigma^{ij}_{\rho_3}},\dots,X_{\sigma^{ij}_{\rho_{k+2}}})\Bigg) \\ &= \frac{1}{2\cdot k!}\sum_{i<j}\Bigg(\sum_{\rho\in S_{k+2},\rho_1=1,\rho_2=2}(-1)^{i+j}\operatorname{sgn}\rho R^\nabla(X_i,X_j)\operatorname{sgn}\rho\,\omega(X_{\sigma^{ij}_3},\dots,X_{\sigma^{ij}_{k+2}}) \end{aligned}$$

$$+\sum_{\rho\in S_{k+2},\rho_1=2,\rho_2=1}(-1)^{i+j}\operatorname{sgn}\rho R^\nabla(X_j,X_i)(-\operatorname{sgn}\rho)\,\omega(X_{\sigma^{ij}_{\rho_3}},\dots,X_{\sigma^{ij}_{\rho_{k+2}}})\Bigg)$$
$$=\sum_{i<j}(-1)^{i+j}R^\nabla(X_i,X_j)\omega(X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+2})$$

□

Lemma 6.6. *Let* $E\to M$ *be a vector bundle,* $p\in M$*,* $\tilde\psi\in E_p$*,* $A\in\operatorname{Hom}(T_pM,E_p)$*. Then there is* $\psi\in\Gamma(E)$ *such that* $\psi_p=\tilde\psi$ *and* $\nabla_X\psi=A(X)$ *for all* $X\in T_pM$*.*

Proof. Choose a frame field $\varphi_1,\dots,\varphi_k$ of E near p. Then we have near p

$$\nabla_X\varphi_i=\sum_{j=1}^k\alpha_{ij}(X)\varphi_j,\quad \textit{for } \alpha_{ij}\in\Omega^1(M),$$

$$A(X)=\sum_{i=1}^k\beta_i\varphi_{ip}\quad \textit{for } \beta\in(T_pM)^*,$$

$$\hat\psi=\sum_{i=1}^k a_i\varphi_{ip}.$$

Ansatz: $\psi=\sum_i f_i\varphi_i$ near $p\rightsquigarrow$ requirements on f_i. Certainly $f_i(p)=a_i$. Further, for $X\in T_pM$,

$$\sum\beta_i(X)\varphi_{ip}=\nabla_X\psi=\sum_i\Big(df_i(X)\varphi_{ip}+f_i(p)\sum_j\alpha_{ij}(X)\varphi_{jp}\Big).$$

With $f_i(p)=a_i$,

$$\beta_i=df_i+\sum_j a_j\alpha_{ji}.$$

Such f_i are easy to find.

□

Theorem 6.7 (Second Bianchi identity)**.** *Let* E *be a vector bundle with connection* ∇*. Then its curvature tensor* $R^\nabla\in\Omega^2(M,\operatorname{End}(E))$ *satisfies*

$$d^\nabla R^\nabla=0.$$

Proof 1. By the last two lemmas we can just choose $X_0,X_1,X_2\in\Gamma(TM)$ commuting near p and $\psi\in\Gamma(E)$ with $\nabla_X\psi=0$ for all $X\in T_pM$. Then near p

$$R^\nabla(X_i,X_j)\psi=\nabla_{X_i}\nabla_{X_j}\psi-\nabla_{X_j}\nabla_{X_i}\psi$$

and thus

$$\begin{aligned}[d^\nabla R^\nabla(X_0,X_1,X_3)]\psi&=(\nabla_{X_0}R^\nabla(X_1,X_2))\psi+(\nabla_{X_1}R^\nabla(X_2,X_0))\psi+(\nabla_{X_2}R^\nabla(X_0,X_1))\psi\\&=\nabla_{X_0}\nabla_{X_1}\nabla_{X_2}\psi-\nabla_{X_0}\nabla_{X_2}\nabla_{X_1}\psi+\nabla_{X_1}\nabla_{X_2}\nabla_{X_0}\psi\\&\quad-\nabla_{X_1}\nabla_{X_0}\nabla_{X_2}\psi+\nabla_{X_2}\nabla_{X_0}\nabla_{X_1}\psi-\nabla_{X_2}\nabla_{X_1}\nabla_{X_0}\psi\\&=R^\nabla(X_0,X_1)\nabla_{X_2}\psi+R^\nabla(X_2,X_0)\nabla_{X_1}\psi+R^\nabla(X_1,X_2)\nabla_{X_0}\psi,\end{aligned}$$

which vanishes at p. □

Proof 2. We have $(d^\nabla R^\nabla)\psi = d^\nabla(R^\nabla\psi) - R^\nabla \wedge d^\nabla\psi = d^\nabla(d^\nabla d^\nabla\psi) - d^\nabla d^\nabla(d^\nabla\psi) = 0.$ □

Exercise 6.8.
Let $M = \mathbb{R}^3$. Determine which of the following forms are closed ($d\omega = 0$) and which are exact ($\omega = d\theta$ for some θ):
a) $\omega = yz\,dx + xz\,dy + xy\,dz,$
b) $\omega = x\,dx + x^2y^2\,dy + yz\,dz,$
c) $\omega = 2xy^2\,dx \wedge dy + z\,dy \wedge dz.$
If ω is exact, please write down the potential form θ explicitly.

Exercise 6.9.
Let $M = \mathbb{R}^n$. For $\xi \in \Gamma(TM)$, we define $\omega^\xi \in \Omega^1(M)$ and $\star\omega^\xi \in \Omega^{n-1}(M)$ as follows:

$$\omega^\xi(X_1) := \langle \xi, X_1\rangle, \quad \star\omega^\xi(X_2,\dots,X_n) := \det(\xi, X_2,\dots,X_n), \quad X_1,\dots,X_n \in \Gamma(TM).$$

Show the following identities:

$$df = \omega^{\mathrm{grad} f}, \quad d\star\omega^\xi = \mathrm{div}(\xi)\det,$$

and for $n = 3$,

$$d\omega^\xi = \star\omega^{\mathrm{rot}\xi}.$$

6.1 Fundamental theorem for flat vector bundles

Let $E \to M$ be a vector bundle with connection ∇. Then

$$E \textit{ trivial} \iff \exists \textit{ frame field } \Phi = (\varphi_1,\dots,\varphi_k) \textit{ with } \nabla\varphi_i = 0,\ i = 1,\dots,k$$

and

$$E \textit{ flat} \iff E \textit{ locally trivial, i.e. each point } p \in M \textit{ has a neighborhood } U \textit{ such that } E|_U \textit{ is trivial.}$$

Theorem 6.10 (Fundamental theorem for flat vector bundles)**.** *A vector bundle*

$$(E,\nabla) \textit{ is flat} \iff R^\nabla = 0.$$

Proof.

"⇒": Let $(\varphi_1,\dots,\varphi_k)$ be a local parallel frame field. Then we have for $i = 1,\dots,k$

$$R^\nabla(X,Y)\varphi_i = \nabla_X\nabla_Y\varphi_i - \nabla_Y\nabla_X\varphi_i - \nabla_{[X,Y]}\varphi_i = 0.$$

Since R^∇ is tensorial checking $R^\nabla\psi = 0$ for the elements of a basis is enough.

"⇐": Assume that $R^\nabla = 0$. Locally we find for each $p \in M$ a neighborhood U diffeomorphic to $(-\varepsilon,\varepsilon)^k$ and a frame field $\Phi = (\varphi_1,\dots,\varphi_k)$ on U. Define $\omega \in \Omega^1(U,\mathbb{R}^{k\times k})$ by

$$\nabla\varphi_i = \sum_{j=1}^k \varphi_j\omega_{ji}.$$

With $\nabla\Phi = (\nabla\varphi_1, \dots, \nabla\varphi_k)$, we write

$$\nabla\Phi = \Phi\omega.$$

Similarly, for a map $F\colon U \to \mathrm{Gl}(k, \mathbb{R})$ define a new frame field:

$$\tilde{\Phi} = \Phi F^{-1}$$

All frame fields on U come from such F. We want to choose F in such a way that $\nabla\tilde{\Phi} = 0$. So,

$$\begin{aligned} 0 &\overset{!}{=} \nabla\tilde{\Phi} \\ &= \nabla(\Phi F^{-1}) \\ &= (\nabla\Phi)F^{-1} + \Phi d(F^{-1}) \\ &= (\nabla\Phi)F^{-1} - \Phi F^{-1} dF\, F^{-1} \\ &= \Phi(\omega - F^{-1}dF)F^{-1}, \end{aligned}$$

where we used that $d(F^{-1}) = -F^{-1}dF\,F^{-1}$. Thus we have to solve

$$dF = F\omega.$$

The *Maurer-Cartan Lemma* (below) states that such $F\colon U \to \mathrm{Gl}(k, \mathbb{R})$ exists if and only if the *integrability condition* (or *Maurer-Cartan equation*)

$$d\omega + \omega \wedge \omega = 0$$

is satisfied. We need to check that in our case the integrability condition holds: We have

$$\begin{aligned} 0 = R^{\nabla}(X,Y)\Phi &= \nabla_X\nabla_Y\Phi - \nabla_Y\nabla_X\Phi - \nabla_{[X,Y]}\Phi \\ &= \nabla_X(\Phi\omega(Y)) - \nabla_Y(\Phi\omega(X)) - \Phi\omega([X,Y]) \\ &= \Phi\omega(X)\omega(Y) + \Phi(X\omega(Y)) - \Phi\omega(Y)\omega(X) - \Phi(Y\omega(X)) - \Phi\omega([X,Y]) \\ &= \Phi(d\omega + \omega\wedge\omega)(X,Y). \end{aligned}$$

Thus $d\omega + \omega \wedge \omega = 0$.

□

Exercise 6.11.
Let $M \subset \mathbb{R}^2$ be open. On $E = M \times \mathbb{R}^2$ we define two connections ∇ and $\tilde{\nabla}$ as follows:

$$\nabla = d + \begin{pmatrix} 0 & -x\,dy \\ x\,dy & 0 \end{pmatrix}, \quad \tilde{\nabla} = d + \begin{pmatrix} 0 & -x\,dx \\ x\,dx & 0 \end{pmatrix}.$$

Show that (E, ∇) is not trivial. Further construct an explicit isomorphism between $(E, \tilde{\nabla})$ and the trivial bundle (E, d).

Lemma 6.12 (Maurer-Cartan)**.** *Let*

$$U := (-\varepsilon, \varepsilon)^n, \quad \omega \in \Omega^1(U, \mathbb{R}^{k\times k}), \quad F_0 \in \mathrm{Gl}(k, \mathbb{R}),$$

then

$$\exists F\colon U \to \mathrm{Gl}(k,\mathbb{R}) : dF = F\omega,\ F(0,\dots,0) = F_0 \quad \Longleftrightarrow \quad d\omega + \omega\wedge\omega = 0$$

Remark 6.13. Note that $d\omega + \omega \wedge \omega$ automatically vanishes on 1-dimensional domains.

Proof.

"⇒": Let $F\colon U \to \mathrm{Gl}(k,\mathbb{R})$ solve the initial value problem $dF = F\omega$, $F(0,\dots,0) = F_0$. Then $0 = d^2F = d(F\omega) = dF \wedge \omega + Fd\omega = F\omega \wedge \omega + Fd\omega = F(d\omega + \omega \wedge \omega)$. Thus $d\omega + \omega \wedge \omega = 0$.

"⇐": (Induction on n) Let $n = 1$. We look for $F\colon (-\varepsilon,\varepsilon) \to \mathrm{Gl}(k,\mathbb{R})$ with

$$dF = F\omega, \quad F(0,\dots,0) = F_0 \in \mathrm{Gl}(k,\mathbb{R}).$$

With $\omega = A\,dx$, this becomes just the linear ODE

$$F' = FA,$$

which is solvable. Only thing still to check that $F(x) \in \mathrm{Gl}(k,\mathbb{R})$ for initial value $F_0 \in \mathrm{Gl}(k,\mathbb{R})$.
But for a solution F we get

$$(\det F)' = (\det F)\,\mathrm{tr}A.$$

Thus if $(\det F)(0) = \det F_0 \neq 0$ then $\det F(x) \neq 0$ for all $x \in (-\varepsilon,\varepsilon)$.
Now let $n > 1$ and suppose that the Maurer-Cartan lemma holds for $n-1$. Write

$$\omega = A_1\,dx_1 + \dots + A_n dx_n$$

with $A_i\colon (-\varepsilon,\varepsilon)^n \to R^{k\times k}$. Then

$$\begin{aligned}(d\omega + \omega\wedge\omega)(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}) &= \Big(\sum_\alpha dA_\alpha \wedge dx_\alpha + \sum_{\alpha,\beta} A_\alpha A_\beta dx_\alpha \wedge dx_\beta\Big)(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j})\\ &= \tfrac{\partial A_j}{\partial x_i} - \tfrac{\partial A_i}{\partial x_j} + A_iA_j - A_jA_i.\end{aligned}$$

By induction hypothesis there is

$$\hat{F}\colon (-\varepsilon,\varepsilon)^{n-1} \to \mathrm{Gl}(k,\mathbb{R})$$

with

$$\tfrac{\partial \hat{F}}{\partial x_i} = \hat{F}A_i$$

for $i = 1,\dots,n-1$, and $\hat{F}(0) = F_0$. Now we solve for each $(x_1,\dots,x_{n-1})$ the initial value problem

$$\begin{aligned}\tilde{F}'_{x_1,\dots,x_{n-1}}(x_n) &= \tilde{F}_{x_1,\dots,x_{n-1}}(x_n)A_n(x_1,\dots,x_n),\\ \tilde{F}_{x_1,\dots,x_{n-1}}(0) &= \hat{F}(x_1,\dots,x_{n-1}).\end{aligned}$$

Define $F(x_1,\dots,x_n) := \tilde{F}_{x_1,\dots,x_{n-1}}(x_n)$. By construction $\frac{\partial F}{\partial x_n} = FA_n$ and with $d\omega + \omega \wedge \omega = 0$,

$$\begin{aligned}\tfrac{\partial}{\partial x_n}(\tfrac{\partial F}{\partial x_i} - FA_i) &= \tfrac{\partial}{\partial x_i}\tfrac{\partial F}{\partial x_n} - \tfrac{\partial}{\partial x_n}(FA_i) = \tfrac{\partial}{\partial x_i}(FA_n) - \tfrac{\partial}{\partial x_n}(FA_i)\\ &= \tfrac{\partial F}{\partial x_i}A_n - \tfrac{\partial F}{\partial x_n}A_i + F(\tfrac{\partial A_n}{\partial x_i} - \tfrac{\partial A_i}{\partial x_n})\\ &= \tfrac{\partial F}{\partial x_i}A_n - FA_nA_i + F(A_nA_i - A_iA_n)\\ &= F(\tfrac{\partial F}{\partial x_i} - FA_i)A_n.\end{aligned}$$

Thus

$$t \mapsto (\tfrac{\partial F}{\partial x_i} - FA_i)(x_1, \ldots, x_{n-1}, t)$$

solves a linear ODE. Since $\frac{\partial F}{\partial x_i} - FA_i = 0$ on the slice $\{x \in (-\varepsilon, \varepsilon)^n \mid x_n = 0\}$, we conclude $\frac{\partial F}{\partial x_\alpha} - FA_\alpha$ for all $\alpha \in \{1, \ldots, n\}$ on whole of $(-\varepsilon, \varepsilon)^n$.

□

Exercise 6.14.
Let $\mathrm{M} \subset \mathbb{R}$ be an interval and consider the vector bundle $\mathrm{E} = \mathrm{M} \times \mathbb{R}^k$, $k \in \mathbb{N}$, equipped with some connection ∇. Show that (E, ∇) is trivial. Furthermore, show that any vector bundle with connection over an intervall is trivial.

7. Riemannian Geometry

7.1 Affine connections

Definition 7.1. *A connection ∇ on the tangent bundle TM is called an affine connection.*

Special about the tangent bundle is that there exists a canonical 1-form $\omega \in \Omega^1(\mathrm{M}, \mathrm{TM})$, the *tautological form*, given by

$$\omega(X) := X.$$

Definition 7.2 (Torsion tensor)**.** *If ∇ is an affine connection on* M*, the* TM*-valued* 2*-form*

$$T^\nabla := d^\nabla \omega$$

is called the torsion tensor of ∇. The connection ∇ is called torsion-free if $T^\nabla = 0$ where ω is the tautological 1*-form.*

Example 7.3. Let $\mathrm{M} \subset \mathbb{R}^n$ open. Identify TM with $\mathrm{M} \times \mathbb{R}^n$ by setting $(p, X)f = d_p f(X)$. On $\mathrm{M} \times \mathbb{R}$ use the trivial connection: All $X \in \Gamma(\mathrm{M} \times \mathbb{R})$ are of the form $X = (\mathrm{Id}, \hat{X})$ for $\hat{X} \in \mathscr{C}^\infty(\mathrm{M}, \mathbb{R}^n)$.

$$(\nabla_X Y)_p = (p, d_p \hat{Y}(X)).$$

Remark (engineer notation):

$$\nabla_X Y = (X \cdot \nabla) Y$$

with $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})^t$ and $X = (x_1, x_2, x_3)$

$$X \cdot \nabla = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$

Define a frame field $X_1, \ldots, X_n$ on M of 'constant vector fields' $X_j = (p, e_j)$. Then with ∇ denoting the trivial connection on $\mathrm{TM} = \mathrm{M} \times \mathbb{R}^n$ we have

$$T^\nabla(X_i, X_j) = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = 0$$

Theorem 7.4 (First Bianchi identity)**.** *Let ∇ be a torsion-free affine connection on* M*. Then for all $X, Y, Z \in \Gamma(\mathrm{TM})$ we have*

$$R^\nabla(X,Y)Z + R^\nabla(Y,Z)X + R^\nabla(Z,X)Y = 0.$$

Proof. For the tautological 1-form $\omega \in \Omega^1(\mathrm{M}, \mathrm{TM})$ and a torsion-free connection we have

$$0 = d^\nabla d^\nabla \omega(X,Y,Z) = R^\nabla \wedge \omega(X,Y,Z) = R^\nabla(X,Y)Z + R^\nabla(Y,Z)X + R^\nabla(Z,X)Y$$

□

Theorem 7.5. *If ∇ is a metric connection on a Euclidean vector bundle $\mathrm{E} \to \mathrm{M}$ then we have for all $X, Y \in \Gamma(\mathrm{TM})$ and $\psi, \varphi \in \Gamma(\mathrm{E})$*

$$\langle R^\nabla(X,Y)\psi, \varphi\rangle = -\langle \psi, R^\nabla(X,Y)\varphi\rangle,$$

i.e. as a 2-form R^∇ takes values in the skew-adjoint endomorphisms.

Proof. The proof is straightforward. We have

$$\begin{aligned} 0 &= d^2\langle\psi,\varphi\rangle \\ &= d\langle d^\nabla\psi, \varphi\rangle + d\langle\psi, d^\nabla\varphi\rangle \\ &= \langle d^\nabla d^\nabla\psi, \varphi\rangle - \langle d^\nabla\psi \wedge d^\nabla\varphi\rangle + \langle d^\nabla\psi \wedge d^\nabla\varphi\rangle + \langle\psi, d^\nabla d^\nabla\varphi\rangle \\ &= \langle d^\nabla d^\nabla\psi, \varphi\rangle + \langle\psi, d^\nabla d^\nabla\varphi\rangle. \end{aligned}$$

With $d^\nabla d^\nabla = R^\nabla$ this yields the statement. □

Definition 7.6 (Riemannian manifold)**.** *A Riemannian manifold is a manifold* M *together with a Riemannian metric, i.e. a metric $\langle .,. \rangle$ on* TM.

Theorem 7.7 (Fundamental theorem of Riemannian geometry)**.** *On a Riemannian manifold there is a unique affine connection ∇ which is both metric and torsion-free. ∇ is called the Levi-Civita connection.*

Proof. Uniqueness: Let ∇ be metric and torsion-free, $X, Y, Z \in \Gamma(\mathrm{TM})$. Then

$$\begin{aligned} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle &= \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle\nabla_Y Z, X\rangle \\ &\quad + \langle Z, \nabla_Y X\rangle - \langle\nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\ &= \langle\nabla_X Y + \nabla_Y X, Z\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle + \langle\nabla_Y Z - \nabla_Z Y, X\rangle \\ &= \langle 2\nabla_X Y - [X,Y], Z\rangle + \langle Y, [X,Z]\rangle + \langle [Y,Z], X\rangle. \end{aligned}$$

Hence we obtain the so called *Koszul formula*:

$$\langle\nabla_X Y, Z\rangle = \frac{1}{2}\big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle + \langle [X,Y],Z\rangle - \langle Y,[X,Z]\rangle - \langle [Y,Z],X\rangle\big).$$

So ∇ is unique. Conversely define $\nabla_X Y$ by the Koszul formula (for this to make sense we need to check tensoriality). Then check that this defines a metric torsion-free connection. □

Exercise 7.8.
Let (M, g) be a Riemannian manifold and $\tilde{g} = e^{2u}g$ for some smooth function $u \colon \mathrm{M} \to \mathbb{R}$. Show that between the corresponding Levi-Civita connections the following relation holds:

$$\tilde{\nabla}_X Y = \nabla_X Y + du(X)Y + du(Y)X - g(X,Y)\operatorname{grad} u.$$

Here $\operatorname{grad} u \in \Gamma(\mathrm{TM})$ is the vector field uniquely determined by the condition $du(X) = g(\operatorname{grad} u, X)$ for all $X \in \Gamma(\mathrm{TM})$.

Definition 7.9 (Riemannian curvature tensor)**.** *Let* M *be a Riemannian manifold. The curvature tensor* R^{∇} *of its Levi-Civita connection* ∇ *is called the Riemannian curvature tensor.*

Exercise 7.10.
Let $(\mathrm{M}, \langle .,. \rangle)$ be a 2-dimensional Riemannian manifold, R its curvature tensor. Show that there is a function $K \in \mathscr{C}^{\infty}(\mathrm{M})$ such that

$$R(X,Y)Z = K\big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\big), \textit{ for all } X,Y,Z \in \Gamma(\mathrm{TM}).$$

Exercise 7.11.
Let $\langle .,. \rangle$ be the Euclidean metric on $\mathbb{R}^n$ and $B := \{x \in \mathbb{R}^n \mid |x|^2 < 1\}$. For $k \in \{-1,0,1\}$ define

$$g_k|_x := \frac{4}{(1+k|x|^2)^2}\langle .,. \rangle.$$

Show that for the curvature tensors R_k of the Riemannian manifolds (B, g_{-1}), $(\mathbb{R}^n, g_0)$ and $(\mathbb{R}^n, g_1)$ and for every $X, Y \in \mathbb{R}^n$ the following equation holds:

$$g_k(R_k(X,Y)Y,X) = k\big(g_k(X,X)g_k(Y,Y) - g_k(X,Y)^2\big).$$

7.2 Flat Riemannian manifolds

The Maurer-Cartan-Lemma states that if $\mathrm{E} \to \mathrm{M}$ is a vector bundle with connection ∇ such that $R^{\nabla} = 0$ then E is flat, i.e. each $p \in \mathrm{M}$ has a neighborhood U and a frame field

$$\varphi_1, \dots, \varphi_k \in \Gamma(\mathrm{E}|_U)$$

with

$$\nabla\varphi_j = 0, \quad j = 1, \dots, k.$$

In fact if we look at the proof we see that given a basis $\psi_1, \dots, \psi_k \in \mathrm{E}_p$ the frame $\varphi_1, \dots, \varphi_k$ can be chosen in such a way that $\varphi_j(p) = \psi_j$, $j = 1, \dots, k$.
Suppose E is Euclidean with compatible ∇ then choose $\psi_1, \dots, \psi_k$ to be an orthonormal basis. Then for each $X \in \Gamma(\mathrm{TU})$ we have

$$X\langle \varphi_i, \varphi_j \rangle = 0, \quad i,j = 1, \dots, k,$$

i.e. (assuming that U is connected) $\varphi_1, \dots, \varphi_k$ is an orthonormal frame field:

$$\langle \varphi_i, \varphi_j \rangle(q) = \delta_{ij}$$

for all $q \in U$. We summarize this in the following theorem.

Theorem 7.12. *Every Euclidean vector bundle with flat connection locally admits an orthonormal parallel frame field.*

Intiution: n-dimensional Riemannian manifolds are "curved versions of $\mathbb{R}^n$". $\mathbb{R}^n =$ *"flat space"*.
The curvature tensor R^{∇} measures curvature, i.e. deviation from flatness.

Definition 7.13 (Isometry). *Let* M *and* N *be Riemannian manifolds. Then* $f\colon \mathrm{M} \to \mathrm{N}$ *is called an isometry if for all* $p \in \mathrm{M}$ *the map*

$$d_p f\colon \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\mathrm{N}$$

is an isometry of Euclidean vector spaces.
In other words, f *is a diffeomorphism such that for all* $p \in \mathrm{M}$, $X, Y \in \mathrm{T}_p\mathrm{M}$ *we have*

$$\langle df(X), df(Y)\rangle_{\mathrm{N}} = \langle X, Y\rangle_{\mathrm{M}}.$$

The following theorem states that any Riemannian manifold with curvature $R = 0$ is locally isometric to $\mathbb{R}^n$.

Theorem 7.14. *Let* M *be an n-dimensional Riemannian manifold with curvature tensor* $R = 0$ *and let* $p \in \mathrm{M}$. *Then there is a neighborhood* $U \subset \mathrm{M}$ *of* p, *an open set* $V \subset \mathbb{R}^n$ *and an isometry* $f\colon U \to V$.

Proof. Choose $\tilde{U} \subset \mathrm{M}$ open, $p \in \tilde{U}$ then there is a parallel orthonormal frame field $X_1, \dots, X_N \in \Gamma(\mathrm{T}\tilde{U})$. Now define

$$\mathrm{E} := \mathrm{TM} \oplus (\mathrm{M} \times \mathbb{R}) = \mathrm{TM} \oplus \mathbb{R}.$$

Any $\psi \in \Gamma(\mathrm{E})$ is of the form

$$\psi = \left(\begin{smallmatrix} Y \\ g \end{smallmatrix}\right)$$

with $Y \in \Gamma(\mathrm{TM})$ and $g \in \mathscr{C}^\infty(\mathrm{M})$. Define a connection $\tilde{\nabla}$ on E as follows

$$\tilde{\nabla}_X \left(\begin{smallmatrix} Y \\ g \end{smallmatrix}\right) := \left(\begin{smallmatrix} \nabla_X Y - gX \\ Xg \end{smallmatrix}\right).$$

It is easy to see that $\tilde{\nabla}$ is a connection. Now

$$\begin{aligned} R^{\tilde{\nabla}}(X,Y)\left(\begin{smallmatrix} Z \\ g \end{smallmatrix}\right) &= \tilde{\nabla}_X \left(\begin{smallmatrix} \nabla_Y Z - gY \\ Yg \end{smallmatrix}\right) - \tilde{\nabla}_Y \left(\begin{smallmatrix} \nabla_X Z - gX \\ Xg \end{smallmatrix}\right) - \left(\begin{smallmatrix} \nabla_{[X,Y]} Z - g[X,Y] \\ [X,Y]g \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} \nabla_X \nabla_Y Z - (Xg)Y - g\nabla_X Y - (Yg)X \\ XYg \end{smallmatrix}\right) - \left(\begin{smallmatrix} \nabla_Y \nabla_X Z - (Yg)X - g\nabla_Y X - (Xg)Y \\ YXg \end{smallmatrix}\right) - \left(\begin{smallmatrix} \nabla_{[X,Y]} Z - g[X,Y] \\ [X,Y]g \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} R(X,Y)Z \\ 0 \end{smallmatrix}\right) = 0. \end{aligned}$$

Now choose $\hat{U} \subset \tilde{U}$, $p \in \hat{U}$ and $\psi \in \Gamma(\mathrm{E}|_{\hat{U}})$ with $\psi_p = (0,1)$, $\tilde{\nabla}\psi = 0$.
Then $\psi = (Y, g)$ with $Y = \sum_{j=1}^{n} f_j X_j$ and

$$\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \nabla_X Y - gX \\ Xg \end{smallmatrix}\right) = \left(\begin{smallmatrix} \sum_j df_j(X)X_j - gX \\ Xg \end{smallmatrix}\right).$$

In particular, $g = 1$. If we define $f\colon \hat{U} \to \mathbb{R}^n$ by $f = (f_1, \dots, f_n)$ then

$$\langle df(X), df(Z)\rangle = \sum_j \langle df_j(X), df_j(Z)\rangle = \langle gX, gY\rangle = \langle X, Y\rangle.$$

In particular, $d_p f$ is bijective. The inverse function theorem then yields a neighborhood U of p such that $f|_U : U \to V \subset \mathbb{R}^n$ is a diffeomorphism and hence an isometry. ☐

Exercise 7.15.
Let M and $\tilde{M}$ be Riemannian manifolds with Levi-Civita connections ∇ and $\tilde{\nabla}$, respectively. Let $f\colon M \to \tilde{M}$ be an isometry and $X, Y \in \Gamma(M)$. Show that $f_*\nabla_X Y = \tilde{\nabla}_{f_*X} f_*Y$.

Remark 7.16. With the last exercise follows that a Riemannian manifold M has curvature $R = 0$ if and only if it is locally isometric to $\mathbb{R}^n$.

Exercise 7.17.
a) Show that $\langle X, Y\rangle := \frac{1}{2}\text{trace}(\bar{X}^t Y)$ defines a Riemannian metric on $SU(2)$.
b) Show that the left and the right multiplication by a constant g are isometries.
c) Show that $SU(2)$ and the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ (with induced metric) are isometric.
Hint: $SU(2) = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}$.

8. Geodesics

Let M be a Riemannian manifold, ∇ the Levi-Civita connection on TM, $\gamma\colon [a,b] \to \mathrm{M}$, $Y \in \Gamma(\gamma^*\mathrm{TM})$.
Then, for $t \in [a,b]$ we have $Y_t \in (\gamma^*\mathrm{TM})_t = \{t\} \times \mathrm{T}_{\gamma(t)}\mathrm{M} \cong \mathrm{T}_{\gamma(t)}\mathrm{M}$. Y is called a *vector field along γ*.
Now define

$$(Y')_t = (\gamma^*\nabla)_{\frac{\partial}{\partial s}\big|_t} Y =: \tfrac{dY}{ds}(t)$$

Definition 8.1 (Geodesic). *$\gamma\colon [a,b] \to \mathrm{M}$ is called a geodesic if $\gamma'' = 0$.*

Exercise 8.2.
Let $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ and $g\colon \tilde{\mathrm{M}} \to \hat{\mathrm{M}}$ be smooth. Show that $f^*(g^*\mathrm{T}\hat{\mathrm{M}}) \cong (g \circ f)^*\mathrm{T}\hat{\mathrm{M}}$ and

$$(g \circ f)^*\hat{\nabla} = f^*(g^*\hat{\nabla})$$

for any affine connection $\hat{\nabla}$ on $\hat{\mathrm{M}}$. Show further that, if f is an isometry between Riemannian manifolds, γ is curve in M and $\tilde{\gamma} = f \circ \gamma$, then

$$\tilde{\gamma}'' = df(\gamma'').$$

Exercise 8.3.
Let M be a Riemannian manifold, $\gamma\colon I \to \mathrm{M}$ be a curve which is parametrized with constant speed, and $f\colon M \to M$ be an isometry which fixes γ, i.e. $f \circ \gamma = \gamma$. Furthermore, let

$$\ker(\mathrm{id} - d_{\gamma(t)}f) = \mathbb{R}\dot{\gamma}(t), \textit{ for all } t.$$

Then γ is a geodesic.

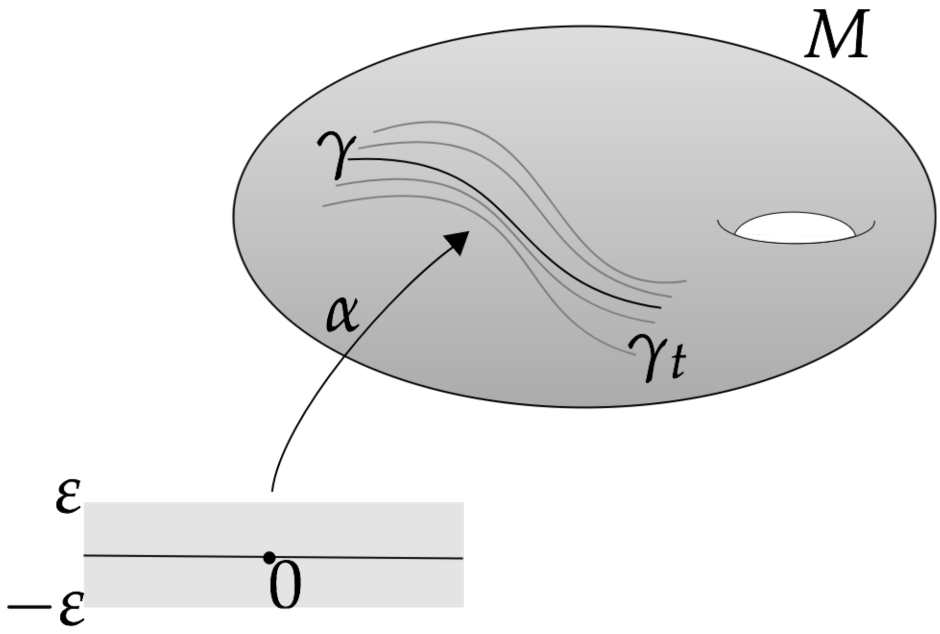
Definition 8.4 (Variation). *A variation of $\gamma\colon [a,b] \to \mathrm{M}$ is a smooth map*

$$\alpha\colon (-\varepsilon,\varepsilon) \times [a,b] \to \mathrm{M}$$

such that $\gamma_0 = \gamma$, where $\gamma_t\colon [a,b] \to \mathrm{M}$ such that $\gamma_t(s) = \alpha(t,s)$. The vector field along γ given by

$$Y_s := \frac{d}{dt}\bigg|_{t=0} \alpha(t,s)$$

is called the variational vector field of α.



Definition 8.5 (Length and energy of curves). *Let* $\gamma\colon [a,b] \to \mathrm{M}$ *be a smooth curve. Then*

$$L(\gamma) := \int_a^b |\gamma'| \text{ is called the length of } \gamma,$$
$$E(\gamma) := \tfrac{1}{2}\int_a^b |\gamma'|^2 \text{ is called the energy of } \gamma.$$

Theorem 8.6. *Let* $\gamma\colon [a,b] \to \mathrm{M}$ *be a smooth curve. Let* $\varphi\colon [c,d] \to [a,b]$ *be smooth with* $\varphi'(t) > 0$ *for all* $t \in [c,d]$, $\varphi(c) = a$ *and* $\varphi(d) = b$. *Then*

$$L(\gamma \circ \varphi) = L(\gamma).$$

Proof. $L(\gamma\circ\varphi) = \int_c^d |(\gamma\circ\varphi)'| = \int_c^d |(\gamma'\circ\varphi)|\varphi' = \int_{\varphi(c)}^{\varphi(d)} |\gamma'| = \int_a^b |\gamma'| = L(\gamma).$ □

Theorem 8.7. *The following inequality holds (with equality if and only if* $|\gamma'|$ *is constant)*

$$E(\gamma) \geq \tfrac{1}{2(b-a)} L(\gamma)^2$$

Proof. The Cauchy-Schwarz inequality yields

$$L(\gamma)^2 \leq 2E(\gamma)\int_a^b 1 = 2(b-a)E(\gamma).$$

□

Theorem 8.8. *Let* $\gamma\colon [a,b] \to \mathrm{M}$ *be a smooth curve such that* $\gamma'(t) \neq 0$ *for all* $t \in [a,b]$. *Then there is a smooth function* $\varphi\colon [0, L(\gamma)] \to [a,b]$ *with* $\varphi'(t) > 0$ *for all* t, $\varphi(0) = a$ *and* $\varphi(L(\gamma)) = b$ *such that* $\tilde\gamma = \gamma\circ\varphi$ *is arclength parametrized, i.e.* $|\tilde\gamma'| = 1$.

Proof. If $\varphi' = 1/|\gamma' \circ \varphi|$, then $|\tilde{\gamma}'| = |(\gamma' \circ \varphi)\varphi'| = 1$. Define $\psi \colon [a,b] \to [0, L(\gamma)]$ by $\psi(t) = \int_a^t |\gamma'|$. Then $\psi'(t) > 0$ for all t, $\psi(a) = 0$ and $\psi(b) = L(\gamma)$. Now set $\varphi = \psi^{-1}$. Then $\varphi' = 1/|\gamma' \circ \varphi|$. □

Theorem 8.9. *Let* $\tilde{\mathrm{M}}$ *be a manifold with torsion-free connection* $\tilde{\nabla}$. *Let* $f \colon \mathrm{M} \to \tilde{\mathrm{M}}$ *and let* $\tilde{\nabla} = f^*\nabla$ *be the pullback connection on* $f^*\mathrm{T}\tilde{\mathrm{M}}$. *Then, if* $X, Y \in \Gamma(\mathrm{TM})$ *we have*

$$df(X), df(Y) \in \Gamma(f^*\mathrm{TM})$$

and

$$\tilde{\nabla}_X df(Y) - \tilde{\nabla}_Y df(X) = df([X,Y])$$

Proof. Let ω denote the tautological 1-form on $\mathrm{T}\tilde{\mathrm{M}}$. Then

$$d^{\tilde{\nabla}}\omega = T^{\tilde{\nabla}} = 0 \quad \text{and} \quad f^*\omega = df.$$

Thus

$$0 = f^* d^{\tilde{\nabla}}\omega = d^{\nabla} f^*\omega = d^{\nabla} df.$$

Thus $0 = d^{\nabla} df(X,Y) = \nabla_X df(Y) - \nabla_Y df(X) - df([X,Y])$. □

Example 8.10. Let $\mathrm{M} \subset \mathbb{R}^n$ be open, and consider the vector fields

$$X = \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \frac{\partial}{\partial x_j}.$$

Then

$$df(X) = \frac{\partial f}{\partial x_i}, \qquad df(Y) = \frac{\partial f}{\partial x_j}$$

and we have

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$$

Hence

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial f}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial f}{\partial x_i}.$$

Theorem 8.11 (First variational formula for energy)**.** *Suppose* $\alpha \colon (-\varepsilon, \varepsilon) \times [a,b] \to \mathrm{M}$ *is a variation of* $\gamma \colon [a,b] \to \mathrm{M}$ *with variational vector field* $Y \in \Gamma(\gamma^*\mathrm{TM})$. *Then*

$$\left.\frac{d}{dt}\right|_{t=0} E(\gamma_t) = \langle Y, \gamma' \rangle \big|_a^b - \int_a^b \langle Y, \gamma'' \rangle.$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E(\gamma_t) &= \frac{d}{dt}\Big|_{t=0} \frac{1}{2}\int_a^b |\gamma_t'|^2 \\
&= \frac{1}{2}\int_a^b \frac{d}{dt}\Big|_{t=0} |\tfrac{\partial\alpha}{\partial s}|^2 \\
&= \int_a^b \left\langle (\alpha^*\nabla)_{\frac{\partial}{\partial t}} \frac{\partial\alpha}{\partial s}\Big|_{(0,s)}, \frac{\partial\alpha}{\partial s} \right\rangle \\
&= \int_a^b \left\langle (\alpha^*\nabla)_{\frac{\partial}{\partial s}} \frac{\partial\alpha}{\partial t}\Big|_{(0,s)}, \frac{\partial\alpha}{\partial s} \right\rangle \\
&= \int_a^b \frac{\partial}{\partial s}\Big|_{(0,s)} \left\langle \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s} \right\rangle - \int_a^b \left\langle \frac{\partial\alpha}{\partial t}, (\alpha^*\nabla)_{\frac{\partial}{\partial s}} \frac{\partial\alpha}{\partial s}\Big|_{(0,s)} \right\rangle \\
&= \int_a^b \frac{d}{ds}\langle Y, \gamma'\rangle - \int_a^b \langle Y, \gamma''\rangle \\
&= \langle Y, \gamma'\rangle\big|_a^b - \int_a^b \langle Y, \gamma''\rangle .
\end{aligned}$$

□

Corollary 8.12. *If α is a variation of γ with fixed endpoints, i.e. $\alpha(t,a) = \gamma(a)$ and $\alpha(t,b) = \gamma(b)$ for all $t \in (-\varepsilon, \varepsilon)$, and γ is a geodesic, then*

$$\frac{d}{dt}\Big|_{t=0} E(\gamma_t) = 0$$

Later we will see the converse statement: If γ is a critical point of E, then γ is a geodesic.

Existence of geodesics:

Let ∇ be an affine connection on an open submanifold $M \subset \mathbb{R}^n$ and let

$$X_i := \tfrac{\partial}{\partial x_i}.$$

Then there are functions Γ_{ij}^k, called Christoffel symbols of ∇, such that

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$$

Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a smooth curve in M, then

$$\gamma' = \sum_i \gamma_i'(\gamma^* X_i).$$

By definition of $\gamma^*\nabla$,

$$(\gamma^* X_j)' = (\gamma^*\nabla)_{\frac{\partial}{\partial s}} \gamma^* X_j = \nabla_{\gamma'} X_j = \sum_i \gamma_i' \gamma^*(\nabla_{X_i} X_j) = \sum_{i,k} \gamma_i' (\Gamma_{ij}^k \circ \gamma) \gamma^* X_k.$$

Thus γ is a geodesic of ∇ if and only if

$$0 = \gamma'' = \sum_j (\gamma_j'' \gamma^* X_j + \gamma_j' \sum_{i,k} \gamma_i' (\Gamma_{ij}^k \circ \gamma) \gamma^* X_k).$$

Since $\gamma^* X_i$ form a frame field we get n equations:

$$0 = \gamma_k'' + \sum_{i,j} \gamma_i' \gamma_j' \Gamma_{ij}^k \circ \gamma.$$

This is an ordinary differential equation of second order and Picard-Lindelöf assures the existence of solutions.

Theorem 8.13 (First variational formula for length)**.** *Let $\gamma\colon [0, L] \to \mathrm{M}$ be arclength parametrized, i.e. $|\gamma'| = 1$. Let $t \to \gamma_t$ for $t \in (-\varepsilon, \varepsilon)$ be a variation of γ with variational vector field Y. Then*

$$\frac{d}{dt}\Big|_{t=0} L(\gamma_t) = \langle Y, \gamma' \rangle\big|_0^L - \int_0^L \langle Y, \gamma'' \rangle$$

Proof. Almost the same as for the first variational formula for energy. □

Theorem 8.14. *Let $\gamma\colon [a, b] \to \mathrm{M}$ be a geodesic. Then $|\gamma'| = constant$.*

Proof. We have $\langle \gamma', \gamma' \rangle' = 2\langle \gamma', \gamma'' \rangle = 0$. □

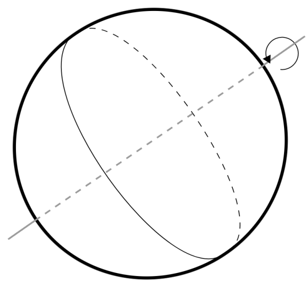


Figure 8.1: A rotation is an isometry on the sphere $\mathbb{S}^2$.

Definition 8.15 (Killing fields)**.** *Suppose $t \to g_t$ for $t \in (-\varepsilon, \varepsilon)$ is a 1-parameter family of isometries of* M*, i.e. each $g_t\colon \mathrm{M} \to \mathrm{M}$ is an isometry. Then the vector field $X \in \Gamma(\mathrm{TM})$,*

$$X_p = \frac{d}{dt}\Big|_{t=0} g_t(p)$$

is called a Killing field of M.

Theorem 8.16. *Let $X \in \Gamma(\mathrm{TM})$ be a Killing field and $\gamma\colon [a, b] \to \mathrm{M}$ be a geodesic, then*

$$\langle X, \gamma' \rangle = constant.$$

Proof. Let $\gamma_t := g_t \circ \gamma$. Then $Y_s = X_{\gamma(s)}$ and $L(\gamma_t) = L(\gamma)$ for all t. Thus

$$0 = \tfrac{d}{dt}\Big|_{t=0} L(\gamma_t) = \langle X_\gamma, \gamma' \rangle\big|_a^b - \int_a^b \langle X_\gamma, \gamma'' \rangle = \langle X_\gamma, \gamma' \rangle\big|_a^b.$$

Thus $\langle X_{\gamma(a)}, \gamma'(a) \rangle = \langle X_{\gamma(b)}, \gamma'(b) \rangle$. □

Example 8.17 (Surface of revolution and Clairaut's relation)**.**
If we have a surface of revolution in Euclidean 3-space, then the rotations about the axis of revolution are isometries of the surface. This yields a Killing field X such that X is orthogonal to the axis of revolution and $|X| = r$, where r denotes the distance to the axis. From the last theorem we know that if γ is a geodesic parametrized with unit speed then $r\cos\alpha = \langle\gamma', X\rangle = c \in \mathbb{R}$. Thus $r = c/\cos\alpha$ and, in particular, $r \geq c$ Thus, depending on the constant c, geodesics cannot pass arbitrarily thin parts.

Example 8.18 (Rigid body motion)**.**
Let $\mathrm{M} = \mathrm{SO}(3) \subset \mathbb{R}^{3\times 3}$, $q_1, \dots, q_n \in \mathbb{R}^3$, $m_1, \dots, m_n > 0$. Now if $t \to A(t)$, $t \in (-\varepsilon, \varepsilon)$, $B = A(0)$, $X = A'(0)$. Then define

$$\langle X, X\rangle = \tfrac{1}{2}\sum_{i=1}^{n} m_i |Xq_i|^2,$$

where $Xq_i = \frac{d}{dt}\Big|_{t=0}(A(t)q_i)$. $\langle X, X\rangle$ is called the kinetic energy at time 0 of the rigid body that undergoes the motion $t \to A(t)$. The principle of least action then says: When no forces act on the body, it will move according to $s \to A(s) \in \mathrm{SO}(3)$ which is a geodesic. For all $G \in \mathrm{SO}(3)$ the left multiplication $A \mapsto GA$ is an isometry. Suitable families $t \mapsto G_t$ with $G_0 = I$ then yields the conservation of angular momentum. We leave the details as exercise.

Theorem 8.19 (Rope construction of spheres)**.** *Given $p \in \mathrm{M}$ and a smooth family, $t \in [0,1]$, of geodesics*

$$\gamma_t \colon [0,1] \to \mathrm{M} \quad \textit{such that} \quad \gamma_t(0) = p$$

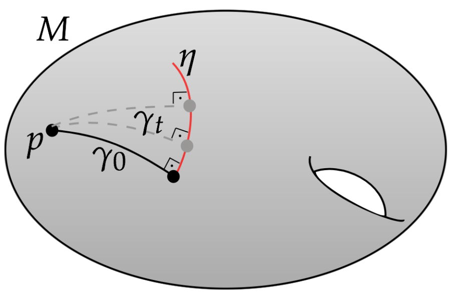
for all t. Let $X(t) \in \mathrm{T}_p\mathrm{M}$ such that

$$X(t) = \gamma_t'(0), \qquad |X| = v \in \mathbb{R},$$

$$\eta\colon [0,1] \to \mathrm{M}, \qquad \eta(t) = \gamma_t(1).$$

Then for all t we have

$$\langle \eta'(t), \gamma_t'(1)\rangle = 0.$$



Proof. Apply the first variational formula to $\gamma = \gamma_t$: Then we have $Y_0 = 0$ and $Y_1 = \eta'$. Since

$$L(\gamma_t) = \int_0^1 |\gamma'| = \int_0^1 |X(0)| = v$$

we have

$$0 = \frac{d}{dt}\Big|_{t=t_0} L(\gamma_t) = \langle \eta'(t_0), \gamma_{t_0}'(1)\rangle - \langle 0, \gamma_{t_0}'(0)\rangle = \langle \eta'(t_0), \gamma_{t_0}'(1)\rangle.$$

□

8.1 The exponential map

Theorem 8.20. *For each $p \in \mathrm{M}$ there is a neighborhood $U \subset \mathrm{M}$ and $\varepsilon > 0$ such that for all $X \in \mathrm{T}_q\mathrm{M}$, $q \in U$, with $|X| < \varepsilon$ there is a geodesic $\gamma\colon [0,1] \to \mathrm{M}$ such that*

$$\gamma(0) = q, \qquad \gamma'(0) = X.$$

Proof. Picard-Lindelöf yields a neighborhood $\tilde{W} \subset \mathrm{TM}$ of $0 \in \mathrm{T}_p\mathrm{M}$ and $\varepsilon_1 > 0$ such that for $X \in \tilde{W}$, $X \in \mathrm{T}_q\mathrm{M}$, there is a geodesic $\gamma\colon [-\varepsilon_1, \varepsilon_1] \to \mathrm{M}$ such that $\gamma(0) = q$ and $\gamma'(0) = X$. Choose $U \subset \mathrm{M}$ open, $\varepsilon_2 > 0$ such that

$$W := \{X \in \mathrm{T}_q\mathrm{M} \mid q \in U,\ |X| \leq \varepsilon_2\} \subset \tilde{W}$$

Now set $\varepsilon = \varepsilon_1\varepsilon_2$.
Let $q \in U$, $X \in \mathrm{T}_q\mathrm{M}$ with $|X| < \varepsilon$ and define

$$Y := \tfrac{1}{\varepsilon_1}X$$

Then $|Y| < \varepsilon_2$, i.e. $Y \in W \subset \tilde{W}$. Thus there exists a geodesic $\tilde{\gamma}\colon [-\varepsilon_1, \varepsilon_1] \to \mathrm{M}$ with $\tilde{\gamma}'(0) = Y$. Now define

$$\gamma\colon [0,1] \to \mathrm{M} \text{ by } \gamma(s) = \tilde{\gamma}(\varepsilon_1 s)$$

Then γ is a geodesic with $\gamma'(0) = \varepsilon_1\tilde{\gamma}'(0) = \varepsilon_1 Y = X$. □

Definition 8.21 (Exponential map)**.** *Let*

$$\Omega := \{X \in \mathrm{TM} \mid \exists\, \gamma\colon [0,1] \to \mathrm{M} \text{ geodesic with } \gamma'(0) = X\}$$

Define

$$\exp\colon \Omega \to \mathrm{M} \text{ by } \exp(X) = \gamma(1)$$

where $\gamma\colon [0,1] \to \mathrm{M}$ is the geodesic with $\gamma'(0) = X$.

Lemma 8.22. *If $\gamma\colon [0,1] \to \mathrm{M}$ is a geodesic with $\gamma'(0) = X$ then $\gamma(t) = \exp(tX)$ for all $t \in [0,1]$.*

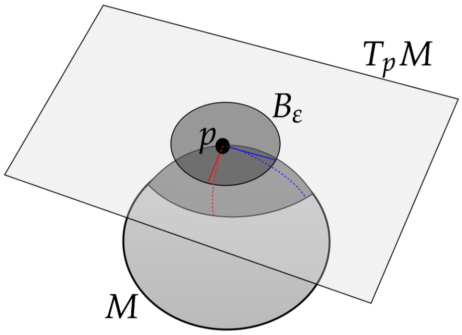
Proof. For $t \in [0,1]$ define $\gamma_t\colon [0,1] \to \mathrm{M}$ by $\gamma_t(s) = \gamma(ts)$. Then $\gamma_t'(0) = tX$, $\gamma_t(1) = \gamma(t)$, γ_t is a geodesic. So $\exp(tX) = \gamma_t(1) = \gamma(t)$. □

Exercise 8.23.
Show that two isometries $F_1, F_2\colon \mathrm{M} \to \mathrm{M}$ which agree at a point p and induce the same linear mapping from T_pM agree on a neighborhood of p.

Theorem 8.24. *Let $p \in \mathrm{M}$. Then there is $\varepsilon > 0$ and an open neighborhood $U \subset \mathrm{M}$ of p such that*

$$B_\varepsilon := \{X \in \mathrm{T}_p\mathrm{M} \mid |X| < \varepsilon\} \subset \Omega \text{ and } \exp|_{B_\varepsilon} : B_\varepsilon \to U$$

is a diffeomorphism.



Proof. From the last lemma we get $d_{0_p}\exp(X) = X$. Here we used the canonical identification between T_pM and $T_{0_p}(TM)$ given by $X \mapsto (t \mapsto tX)$. The claim then follows immediately from the inverse function theorem. □

Definition 8.25 (Geodesic normal coordinates).

$$(\exp|_{B_\varepsilon})^{-1} \colon U \to B_\varepsilon \subset T_pM \cong \mathbb{R}^n$$

viewed as a coordinate chart is called geodesic normal coordinates near p.

Exercise 8.26.
Let M be a Riemannian manifold of dimension n. Show that for each point $p \in M$ there is a local coordinate $\varphi = (x_1, \dots, x_n)$ at p such that

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\Big|_p = \delta_{ij}, \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\Big|_p = 0.$$

Theorem 8.27 (Gauss lemma). $\exp|_{B_\varepsilon}$ *maps radii* $t \mapsto tX$ *in* B_ε *to geodesics in* M. *Moreover, these geodesics intersect the hypersurfaces* $S_r := \{\exp(X) \mid X \in B_\varepsilon, |X| = r\}$ *orthogonally.*

Proof. This follows by the last lemma and the rope construction of spheres. □

Definition 8.28 (Distance). *Let* M *be a connected Riemannian manifold. Then for* $p, q \in M$ *define the distance* $d(p,q)$ *by*

$$d(p,q) = \inf\{L(\gamma) \mid \gamma \colon [0,1] \to M \text{ smooth with } \gamma(0) = p,\ \gamma(1) = q\}.$$

Exercise 8.29.

a) Is there a Riemannian manifold (M, g) which has finite diameter (i.e. there is an m such that all points $p, q \in M$ have distance $d(p,q) < m$) and there is a geodesic of infinite length without self-intersections?
b) Find an example for a Riemannian manifold diffeomorphic to $\mathbb{R}^n$ but which has no geodesic of infinite length.

Definition 8.30 (Metric space). *A metric space is a pair* (X, d) *where* X *is a set and*

$$d\colon X \times X \to \mathbb{R}$$

a map such that

a) $d(p,q) \geq 0$, $d(p,q) = 0 \Leftrightarrow p = q$,

b) $d(p,q) = d(q,p)$,

c) $d(p,q) + d(q,r) \geq d(p,r)$.

Is a Riemannian manifold (with its distance) a metric space?

Symmetry:

Symmetry is easy to see: If $\gamma\colon [0,1] \to M$ is a curve from p to q, then $\tilde{\gamma}(t) := \gamma(1-t)$ is a curve from q to p and $L(\tilde{\gamma}) = L(\gamma)$.

Triangle inequality:

For the triangle inequality we need to concatenate curves. So let $\gamma\colon [0,1] \to M$ be a curve from p to q and $\tilde{\gamma}\colon [0,1] \to M$ be a curve from q to r.
Though the naive concatenation is not smooth we can stop for a moment and then continue running:
Let $\varphi\colon [0,1] \to [0,1]$ be smooth monotone function such that $\varphi(0) = 0$, $\varphi(1) = 1$ and φ' vanishes on $[0,\varepsilon) \cup (1-\varepsilon, 1]$ for some $\varepsilon > 0$ sufficiently small. Then define for γ from p to q and $\tilde{\gamma}$ from q to r

$$\hat{\gamma}(t) = \begin{cases} \gamma(\varphi(2t)), & \textit{for } t \in [0,1/2) \\ \tilde{\gamma}(\varphi(2t-1)), & \textit{for } t \in [1/2,1] \end{cases}$$

Then

$$L(\hat{\gamma}) = L(\gamma) + L(\tilde{\gamma})$$

For every $\varepsilon > 0$ we find γ and $\tilde{\gamma}$ such that

$$L(\gamma) \leq d(p,q) + \varepsilon, \quad L(\tilde{\gamma}) \leq d(q,r) + \varepsilon$$

Thus by concatenation we obtain a curve $\hat{\gamma}$ from q to r such that

$$L(\hat{\gamma}) \leq d(p,q) + d(q,r) + 2\varepsilon$$

Thus

$$d(p,r) \leq d(p,q) + d(q,r)$$

As certainly it holds that $L(\gamma) \geq 0$ this leads to $d(p,q) \geq 0$ and $d(p,p) = 0$.

So the only part still missing is that $p = q$ whenever $d(p,q) = 0$.

Theorem 8.31. *Let* $p \in M$ *and* $f\colon B_\varepsilon \to U \subset M$ *be geodesic normal coordinates at* p. *Then*

$$d(p, \exp(X)) = |X|, \quad \textit{for } |X| \leq \varepsilon.$$

Moreover, for $q \notin U$, $d(p,q) > \varepsilon$.

Proof. Choose $0 < R < \varepsilon$. Take $\gamma\colon [0,1] \to \mathrm{M}$ with $\gamma(0) = p$, and define

$$\gamma(t) := \exp(tX) \text{ with } |X| = R$$

Let

$$q := \gamma(1) = \exp(X).$$

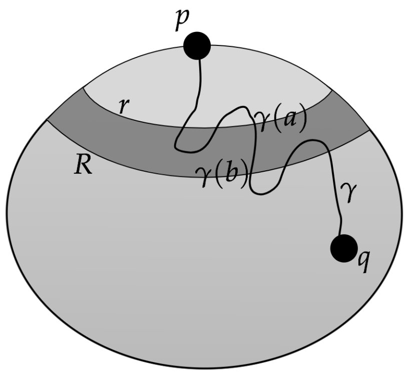
Then $L(\gamma) = R$ and in particular $d(p,q) \leq R$.

Now, choose $0 < r < R$ and let $\gamma\colon [0,1] \to \mathrm{M}$ be any curve with $\gamma(0) = p$ and $\gamma(1) = q$. Define a to be the greatest $t \in [0,1]$ such that there is Y such that

$$\gamma(a) = \exp(Y), \quad |Y| = r.$$

Define b to be the smallest $t \in [0,1]$, $a < b$, such that there is Z such that

$$\gamma(b) = \exp(Z), \quad |Z| = R.$$



Now find

$$\xi\colon [a,b] \to \mathrm{TM}$$

such that

$$r < |\xi(t)| < R \text{ for all } t \in (a,b), \quad |\xi(a)| = r, \quad |\xi(b)| = R$$

and

$$\exp(\xi(t)) = \gamma(t)$$

for all $t \in [a,b]$. Define

$$\rho\colon [a,b] \to \mathbb{R} \quad \text{by} \quad \rho := |\xi|$$

and

$$\nu\colon [a,b] \to \mathbb{S}^{n-1} \subset T_pM \quad \text{by} \quad \xi =: \rho\nu.$$

Claim: It holds that $L(\gamma|_{[a,b]}) \geq R - r$.

Proof. For all $t \in [a,b]$ we have

$$\gamma'(t) = d\exp\big(\xi'(t)\big) = d\exp\big(\rho'(t)\nu(t) + \rho(t)\nu'(t)\big) = \rho'(t)d\exp(\nu(t)) + \rho(t)d\exp\big(\nu'(t)\big)$$

By the Gauss lemma we get then

$$|\gamma'(0)|^2 = |\rho'(t)d\exp(\nu(t))|^2 + |\rho(t)d\exp\big(\nu'(t)\big)|^2 \geq |\rho'(t)|^2 \underbrace{|d\exp(\nu(t))|^2}_{=1} = \rho'(t)^2$$

Thus we have

$$L(\gamma|_{[a,b]}) = \int_a^b |\rho'| \geq \int_a^b \rho' = \rho|_a^b = R - r$$

Certainly, we can have equality only for $\nu' = 0$. This yields the second part. □

Now $L(\gamma) \geq R - r$ for all such $r > 0$. Hence $L(\gamma) \geq R$ and thus $d(p,q) = R$. □

Corollary 8.32. *A Riemannian manifold together with its distance function is a metric space.*

Corollary 8.33. *Let $\gamma\colon [0,L] \to \mathrm{M}$ be an arclength-parametrized geodesic. Then there is $\varepsilon > 0$ such that*

$$d(\gamma(0),\gamma(t)) = t \text{ for all } t \in [0,\varepsilon]$$

The first variational formula says:
If $\gamma\colon [a,b] \to \mathrm{M}$ is a smooth length-minimizing curve, i.e.

$$L(\gamma) = d(\gamma(a),\gamma(b)),$$

then γ is a geodesic.

To see this, choose a function $\rho\colon [a,b] \to \mathbb{R}$ with $\rho(s) > 0$ for all $s \in (a,b)$ but

$$\rho(a) = 0 = \rho(b).$$

Then there is $\varepsilon > 0$ such that

$$\alpha\colon (-\varepsilon,\varepsilon) \times (a,b) \to \mathrm{M},\ \alpha(t,s) = \exp\big(t\rho(s)\gamma''(s)\big).$$

Without loss of generality we can assume that $|\gamma'| = 1$, then

$$0 = \frac{d}{dt}\bigg|_{t=0} L(\gamma) = \underbrace{\langle\gamma',\rho\gamma''\rangle\big|_a^b}_{=0} - \int_a^b \langle\rho\gamma'',\gamma''\rangle = -\int_a^b \rho|\gamma''|^2$$

for all such ρ. Thus we conclude $\gamma'' = 0$ and so γ is a geodesic. We need a slightly stronger result.
For preparation we give the following exercise:

Exercise 8.34.
$d(p,q) = \inf\{L(\gamma) \mid \gamma\colon [a,b] \to \mathrm{M}$ *piecesewise smooth*, $\gamma(a) = p$, $\gamma(b) = q\}$.

Theorem 8.35. *Let $\gamma\colon [0,L] \to \mathrm{M}$ be a continuous piecewise-smooth curve with $|\gamma'| = 1$ (whenever defined) such that $L(\gamma) = d(\gamma(0),\gamma(L))$. Then γ is a smooth geodesic.*

Proof. Let $0 = s_0 < \ldots < s_k = L$ be such that $\gamma|_{[s_{i-1},s_i]}$ is smooth, $i = 1,\ldots,k$. The above discussion then shows that the parts $\gamma|_{[s_{i-1},s_i]}$ are smooth geodesics. We need to show that there are no kinks.
Let $j \in \{1,\ldots,k-1\}$ and

$$X := \gamma'\big|_{[s_{j-1},s_j]}(s_j), \quad \tilde{X} = \gamma'\big|_{[s_j,s_{j+1}]}(s_j)$$

Claim:
It holds that $X = \tilde{X}$.

Proof. Define $Y = \tilde{X} - X$ and choose any variation γ_t of γ which does nothing on $[0,s_{j-1}] \cup [s_{j+1},L]$. Then

$$0 = \frac{d}{dt}\bigg|_{t=0} L(\gamma_t) = \sum_j \frac{d}{dt}\bigg|_{t=0} L(\gamma_t|_{[s_{j-1},s_j]}) = \langle Y,X\rangle - \langle Y,\tilde{X}\rangle = |\tilde{X} - X|^2.$$

Thus $\tilde{X} - X = 0$. □

□

8.2 Complete Riemannian manifolds

Definition 8.36 (Complete Riemannian manifold)**.** *A Riemannian manifold is called complete if* exp *is defined on all of* TM*, or equivalently: every geodesic can be extended to* $\mathbb{R}$*.*

Theorem 8.37 (Hopf and Rinow)**.** *Let* M *be a complete Riemannian manifold,* $p, q \in$ M*. Then there is a geodesic* $\gamma\colon [0, L]$ *with* $\gamma(0) = p$*,* $\gamma(L) = q$ *and* $L(\gamma) = d(p, q)$*.*

Proof. Let $\varepsilon > 0$ be such that $\exp|_{B_\varepsilon}$ is a diffeomorphism onto its image. Without loss of generality, assume that $\delta < d(p, q)$.
Let $0 < \delta < \varepsilon$ and set

$$S := \exp(S_\delta)$$

where $S_\delta = \partial B_\delta$. Then

$$f\colon S \to \mathbb{R} \text{ given by } f(r) = d(r, q)$$

is continuous. Since S is compact, there is $r_0 \in S$ where f has a minimum, i.e.

$$d(r_0, q) \le d(r, p) \textit{ for all } r \in S$$

Then $r_0 = \gamma(\delta)$, where $\gamma\colon \mathbb{R} \to$ M with $\gamma(0) = p$.
Define

$$d(S, q) := \inf\{d(r, q) \mid r \in S\}$$

then

$$d(S, q) = d(r_0, q)$$

Every curve $\eta\colon [a, b] \to$ M from p to q has to hit S: There is $t_0 \in [a, b]$ with $\eta(t_0) \in S$.
Moreover,

$$L(\eta) = L(\eta|_{[a,t_0]}) + L(\eta|_{[t_0,b]}) \ge \delta + d(S, q) = \delta + d(r_0, q)$$

so

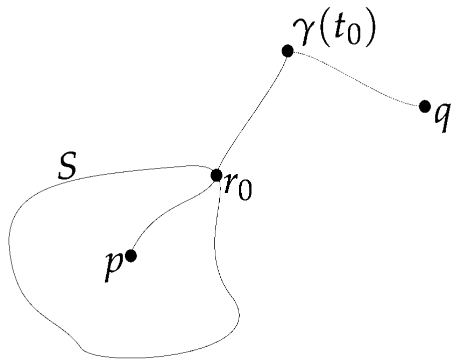
$$d(p, q) \ge \delta + d(r, q)$$

On the other hand, the triangle inequality yields

$$d(p, q) \le d(p, r_0) + d(r_0, q) = \delta + d(r_0, q)$$

thus

$$d(\gamma(\delta), q) = d(p, q) - \delta$$



Define statement $A(t)$: "$d(\gamma(t),q) = d(p,q) - t$"
So we know $A(\delta)$ is true. We want to show that also $A(d(p,q))$ is true.
Define

$$t_0 := \sup\{t \in [0, d(p,q)] \mid A(t) \ true\}.$$

Assume that $t_0 < d(p,q)$.

Claim: $A(t_0)$ is true.

Proof. This is because there is a sequence $t_1, t_2, \ldots,$ with $\lim_{n\to\infty} t_n = t_0$ and $A(t_n)$ true, i.e. $f(t_n) = 0$ where $f(t) = d(\gamma(t),q) - (d(p,q) - t)$. Clearly, f is continuous. Thus $f(t_0) = 0$, too. □

Now let $\tilde{\gamma}$ be a geodesic constructed as before but emanating from $\gamma(t_0)$.
With the same argument as before we then get again

$$d(\tilde{\gamma}(\tilde{\delta}), q) = d(\tilde{\gamma}(0), q) - \tilde{\delta}.$$

Now, since $A(t_0)$ is true, we have

$$d(p,q) \leq d(p, \tilde{\gamma}(\tilde{\delta})) + d(\tilde{\gamma}(\tilde{\delta}), q) = d(p, \tilde{\gamma}(\tilde{\delta})) + d(\tilde{\gamma}(0), q) - \tilde{\delta} = d(p, \tilde{\gamma}(\tilde{\delta})) + d(p,q) - t_0 - \tilde{\delta}$$

There obviously is a piecewise-smooth curve from p to $\tilde{\gamma}(\tilde{\delta})$ of length $t_0 + \tilde{\delta}$, so

$$d(p, \tilde{\gamma}(\tilde{\delta})) \leq t_0 + \tilde{\delta}$$

hence

$$d(p, \tilde{\gamma}(\tilde{\delta})) = t_0 + \tilde{\delta}$$

henceforth this piecewise-smooth curve is length minimizing and in particular it is smooth, i.e. there is no kink and thus we have $\tilde{\gamma}(\tilde{\delta}) = \gamma(t_0 + \tilde{\delta})$.
Now we have

$$d(\gamma(t_0 + \tilde{\delta}), q) = d(\gamma(t_0), q) - \tilde{\delta} = d(p,q) - (t_0 + \tilde{\delta})$$

Thus $A(t_0 + \tilde{\delta})$ is true, which contradicts the definition of t_0. So $A(d(p,q))$ is true. □

Theorem 8.38. *For a Riemannian manifold* M *the following statements are equivalent:*

a) M *is complete Riemannian manifold.*

b) All bounded closed subsets of M *are compact.*

c) (M, d) *is a complete metric space.*

Proof.

a) ⇒ *b)*: Let $A \subset \mathrm{M}$ be closed and bounded, i.e. there is $p \in \mathrm{M}$ and $c \in \mathbb{R}$ such that $d(p,q) \leq c$ for a all $p, q \in A$. Look at the ball $B_c \subset \mathrm{T}_p\mathrm{M}$. Hopf-Rinow implies then that $A \subset \exp(B_c)$. Hence A is a closed subset of a compact set and thus compact itself.

b) ⇒ *c)*: This is a well-known fact: Any Cauchy sequence $\{p_n\}_{n\in\mathbb{N}}$ is bounded and thus lies in bounded closed set which then is compact. Hence $\{p_n\}_{n\in\mathbb{N}}$ has a convergent subsequence which then converges to the limit of $\{p_n\}_{n\in\mathbb{N}}$.

c) $\Rightarrow$ *a*)**:** Let $\gamma\colon [0,\ell] \to \mathrm{M}$ be a geodesic.

$$T := \sup\{t \geq \ell \mid \gamma \textit{ can be extended to } [0,T]\}.$$

We want to show that $T = \infty$. Define $p_n := \gamma(T - \frac{1}{n})$. Then $\{p_n\}_{n\in\mathbb{N}}$ defines a Cauchy sequence which thus has a limit point $p := \lim\limits_{n\to\infty} p_n$. Thus γ extends to $[0,T]$ by setting $\gamma(T) := p$. Thus γ extends beyond T. which contradicts the definition of T.

□

Exercise 8.39. A curve γ in a Riemannian manifold M is called *divergent*, if for every compact set $K \subset \mathrm{M}$ there exists a $t_0 \in [0,a)$ such that $\gamma(t) \notin K$ for all $t > t_0$. Show: M is complete if and only if all divergent curves are of infinite length.

Exercise 8.40. Let M be a complete Riemannian manifold, which is not compact. Show that there exists a geodesic $\gamma\colon [0,\infty) \to \mathrm{M}$ which for every $s > 0$ is the shortest path between $\gamma(0)$ and $\gamma(s)$.

Exercise 8.41. Let M be a compact Riemannian manifold. Show that M has finite diameter, and that any two points $p, q \in \mathrm{M}$ can be joined by a geodesic of length $d(p,q)$.

9. Topics on Riemannian Geometry

9.1 Sectional curvature

Definition 9.1 (Sectional curvature). *Let* M *be a Riemannian manifold,* $p \in \mathrm{M}$, $\mathrm{E} \subset \mathrm{T}_p\mathrm{M}$, $\dim \mathrm{E} = 2$, $\mathrm{E} = \mathrm{span}\{X, Y\}$. *Then*

$$K_{\mathrm{E}} := \frac{\langle R(X,Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}$$

is called the sectional the sectional curvature of E.

Exercise 9.2. Check that K_{E} is well-defined.

Theorem 9.3. *Let* M *be a Riemannian manifold,* $p \in \mathrm{M}$, *and* $X, Y, Z, W \in \mathrm{T}_p\mathrm{M}$. *Then*

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle.$$

Proof. The Jacobi identity yields the following 4 equations:

$$\begin{aligned}
0 &= \langle R(X,Y)Z, W\rangle + \langle R(Y,Z)X, W\rangle + \langle R(Z,X)Y, W\rangle, \\
0 &= \langle R(Y,Z)W, X\rangle + \langle R(Z,W)Y, X\rangle + \langle R(W,Y)Z, X\rangle, \\
0 &= \langle R(W,Z)X, Y\rangle + \langle R(X,W)Z, Y\rangle + \langle R(Z,X)W, Y\rangle, \\
0 &= \langle R(X,W)Y, Z\rangle + \langle R(Y,X)W, Z\rangle + \langle R(W,Y)X, Z\rangle
\end{aligned}$$

□

The following theorem tells us that the sectional curvature completely determine the curvature tensor R.

Theorem 9.4. *Let* V *be a Euclidean vector space.* $R\colon V \times V \to V$ *bilinear with all the symmetries of the curvature tensor of a Riemannian manifold. For any 2-dimensional subspace* $E \subset V$ *with orthonormal basis* X, Y *define*

$$K_E := \langle R(X,Y)Y, X\rangle$$

Let $\tilde{R}$ *be another such tensor with* $\tilde{K}_E = K_E$ *for all 2-dimensional subspaces* $E \subset V$, *then* $\tilde{R} = R$.

Proof. $K_E = \tilde{K}_E$ implies

$$\langle R(X,Y)Y, X\rangle = \langle \tilde{R}(X,Y)Y, X\rangle \text{ for } \textit{all } X, Y \in V$$

We will show that we can calculate $\langle R(X,Y)Z,W\rangle$ for all $X,Y,Z,W \in V$ provided we know $\langle R(X,Y)Y,X\rangle$ for all $X,Y \in V$.
Let $X,Y,Z,W \in V$ and define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(s,t) := \langle R(X+sW,Y+tZ)(Y+tZ),X+sW\rangle - \langle R(X+sZ,Y+tW)(Y+tW),X+sZ\rangle$$

For fixed X,Y,Z,W this is polynomial in s and t.
We are only interested in the st term: It is

$$\begin{aligned}
&\langle R(W,Z)Y,X\rangle + \langle R(W,Y)Z,X\rangle + \langle R(X,Z)Y,W\rangle + \langle R(X,Y)Z,W\rangle \\
&\quad - \langle R(Z,W)Y,X\rangle - \langle R(Z,Y)W,X\rangle - \langle R(X,W)Y,Z\rangle - \langle R(X,Y)Z,W\rangle \\
&= 4\langle R(X,Y)Z,W\rangle + 2\langle R(W,Y)Z,X\rangle - 2\langle R(Z,Y)W,X\rangle \\
&= 4\langle R(X,Y)Z,W\rangle + 2\langle R(W,Y)Z + R(Y,Z)W,X\rangle \\
&= 4\langle R(X,Y)Z,W\rangle - 2\langle R(Z,W)Y,X\rangle \\
&= 6\langle R(X,Y)Z,W\rangle.
\end{aligned}$$

□

Corollary 9.5. *Let* M *be a Riemannian manifold and* $p \in \mathrm{M}$*. Suppose that* $K_E = K$ *for all* $E \subset \mathrm{T}_p\mathrm{M}$ *with* $\dim E = 2$*, then*

$$R(X,Y)Z = K(\langle Z,Y\rangle X - \langle Z,X\rangle Y).$$

Proof. Define $\tilde{R}$ by this formula. Then $\tilde{R}(X,Y)$ is skew in X,Y and

$$\langle \tilde{R}(X,Y)Z,W\rangle = K(\langle Y,Z\rangle\langle X,W\rangle - \langle Z,X\rangle\langle Y,W\rangle)$$

is skew in Z,W. Finally,

$$\begin{aligned}
&\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y \\
&\qquad = K(\langle Z,Y\rangle X - \langle Z,X\rangle Y + \langle X,Z\rangle Y - \langle X,Y\rangle Z + \langle Y,X\rangle Z - \langle Y,Z\rangle X) = 0.
\end{aligned}$$

and if $X,Y \in \mathrm{T}_p\mathrm{M}$ is an orthonormal basis then

$$\tilde{K}_E = K\langle\langle Y,Y\rangle X - \langle Y,X\rangle Y,X\rangle = K.$$

□

9.2 Jacobi fields

Let $\gamma\colon [0,L] \to \mathrm{M}$ be a geodesic and

$$\alpha\colon (-\varepsilon,\varepsilon) \times [0,L] \to \mathrm{M}$$

be a geodesic variation of γ, i.e. $\gamma_t = \alpha(t,.)$ is a geodesic for all $t \in (-\varepsilon,\varepsilon)$. Then the corresponding variational vector field $Y \in \Gamma(\gamma^*\mathrm{TM})$ along γ,

$$Y_s = \left.\frac{\partial\alpha}{\partial t}\right|_{(0,s)}$$

is called a Jacobi field.

Lemma 9.6. *Let α be a variation of a curve, $\tilde{\nabla} = \alpha^*\nabla$ and $\tilde{R} = \alpha^* R$, then*

$$\tilde{\nabla}_{\frac{\partial}{\partial s}}\tilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}\alpha = \tilde{R}(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})\frac{\partial}{\partial s}\alpha + \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}\alpha.$$

Proof. Since ∇ is torsion-free we have $\tilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}\alpha = \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}\alpha$. The equation then follows from $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$. □

Theorem 9.7. *A vector field $Y \in \Gamma(\gamma^*\mathrm{TM})$ is a Jacobi field if and only if it satisfies*

$$Y'' + R(Y, \gamma')\gamma' = 0.$$

Proof.

"⇒": With the lemma above evaluated for $(0, s)$ we obtain

$$Y'' = \tilde{R}(\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t})\tfrac{\partial}{\partial s}\alpha\Big|_{(0,s)} + \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\Big|_{(0,s)} = \tilde{R}(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})\tfrac{\partial}{\partial s}\alpha\Big|_{(0,s)} = R(\gamma', Y)\gamma'$$

"⇐": Suppose a vector field Y along γ satisfies

$$Y'' + R(Y, \gamma')\gamma' = 0$$

We want to construct a geodesic variation α such that $\gamma_0 = \gamma$ and with variational vector field Y. The solution of a linear second order ordinary differential equation Y is uniquely prescribed by $Y(0)$ and $Y'(0)$. In particular the Jacobi fields form a $2n$-dimensional vector space. Denote $p := \gamma(0)$.

By the first part of the prove it is enough to show that for each $V, W \in \mathrm{T}_p\mathrm{M}$ there exists a geodesic variation α of γ with variational vector field Y which satisfies $Y(0) = V$ and $Y'(0) = W$:

The curve

$$\eta\colon (-\tilde{\varepsilon}, \tilde{\varepsilon}) \to \mathrm{M}, \quad \eta(t) = \exp(tV)$$

is defined for $\tilde{\varepsilon} > 0$ small enough.

Define a parallel vector field $\tilde{W}$ along η with $\tilde{W}_0 = W$. Similarly, let $\tilde{U}$ be parallel along η such that $\tilde{U}_0 = \gamma'(0)$.

Now define

$$\alpha\colon [0, L] \times (-\varepsilon, \varepsilon) \to \mathrm{M} \text{ by } \alpha(s, t) = \exp\big(s(\tilde{U}_t + t\tilde{W}_t)\big)$$

for $\varepsilon > 0$ small enough. Clearly, α is a geodesic variation of γ.

From $\tilde{U}_t, \tilde{W}_t \in \mathrm{T}_{\eta(t)}\mathrm{M}$ we get

$$\gamma_t(0) = \eta(t)$$

and hence

$$Y(0) = \eta'(0) = V$$

Moreover,

$$Y'(0) = \dot{\alpha}'(0,0) = \nabla_{\frac{\partial}{\partial t}|_{(s,t)=(0,0)}}\alpha' = \nabla_{\frac{\partial}{\partial t}|_{t=0}}(\tilde{U}_t + t\tilde{W}_t) = \tilde{W}_0 = W$$

□

Exercise 9.8.
Show that, as claimed in the previous proof, there is $\varepsilon > 0$ such that for $|t| < \varepsilon$ the geodesic $\gamma_t = \alpha(.,t)$ really lives for time L.

Trivial geodesic variations: $\gamma_t(s) = \gamma(a(t)s + b(t))$ with functions a and b such that $a(0) = 1$, $b(0) = 0$. Then the variational vector field is just

$$Y_s = (a'(0)s + b'(0))\gamma'(s)$$

Thus $Y' = a'(0)\gamma'$ and hence $Y'' = 0$. Certainly also $R(Y,\gamma') = 0$, thus Y is a Jacobi field.
Interesting Jacobi fields are orthogonal to γ':
Let Y be a Jacobi-field, then consider

$$f\colon [0,L] \to \mathbb{R}, \quad f = \langle Y, \gamma' \rangle$$

Then

$$f' = \langle Y', \gamma' \rangle \text{ and } f'' = \langle Y'', \gamma' \rangle = -\langle R(Y,\gamma')\gamma', \gamma' \rangle = 0$$

Thus there are $a, b \in \mathbb{R}$ such that

$$f(s) = as + b$$

In particular, with

$$V := Y(0) \text{ and } W := Y'(0)$$

we have

$$f(0) = \langle V, \gamma' \rangle, \quad f'(0) = \langle W, \gamma'(0) \rangle$$

Then we will have $f \equiv 0$ provided that $V, W \perp \gamma'(0)$. So $\langle Y, \gamma' \rangle \equiv 0$ in this case. This defines a $(2n-2)$-dimensional space of (interesting) Jacobi fields.

Example 9.9. Consider $\mathrm{M} = \mathbb{R}^n$. Then Y Jacobi field along $s \mapsto p + sv$ if and only if $Y'' \equiv 0$, i.e. $Y(s) = V + sW$ for parallel vector fields V, W along γ (constant).

Example 9.10. Consider the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and let $p, V, W \in \mathbb{R}^{n+1}$ be orthonormal. Define γ_t as follows

$$\gamma_t(s) := \cos s\, p + \sin s\, (\cos t\, V + \sin t\, W).$$

Then $Y_s = \sin s\, W$ is a Jacobi field and thus

$$-\sin s\, W = Y''(s) = -R(Y(s), \gamma'(0))\gamma'(0) = -\sin s R(W, \gamma')\gamma'.$$

Thus $W = R(W,\gamma')\gamma'$. Evaluation for $s = 0$ then yields $W = R(W,V)V$. In particular, if $E = \mathrm{span}\{V, W\} \subset \mathrm{T}_p\mathbb{S}^n$, then $K_E = \langle R(W,V)V, W \rangle = 1$.

9.3 Second variational formula

Theorem 9.11 (Second variational formula). *Let*

$$\alpha\colon (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times[0,L]\to \mathrm{M}$$

be a 2-parameter variation of a geodesic $\gamma\colon [0,L]\to \mathrm{M}$*, i.e.* $\alpha(0,0,s)=\gamma(s)$*, with fixed endpoints, i.e.* $\alpha(u,v,0)=\gamma(0)$ *and* $\alpha(u,v,L)=\gamma(L)$ *for all* $u,v\in(-\varepsilon,\varepsilon)$*. Let*

$$X_s = \frac{\partial\alpha}{\partial u}\bigg|_{(0,0,s)} \qquad Y_s = \frac{\partial\alpha}{\partial v}\bigg|_{(0,0,s)} \qquad \gamma_{u,v}(s) := \alpha(u,v,s)$$

Then

$$\frac{\partial^2}{\partial u\partial v}E(\gamma_{u,v})(0,0) = -\int_0^L \langle X, Y'' + R(Y,\gamma')\gamma'\rangle.$$

Remark 9.12. Actually, that is an astonishing formula. Since the left hand side is symmetric in u and v, the right hand side must be symmetric in X and Y.
Let's check this first directly:
Let $X,Y\in\Gamma(\gamma^*\mathrm{TM})$ such that $X_0=0=Y_0$ and $X_L=0=Y_L$. Then with partial integration we get

$$\begin{aligned}\int_0^L \langle X, Y''+R(Y,\gamma')\gamma'\rangle &= \int_0^L\langle X,Y''\rangle + \int_0^L \langle X, R(Y,\gamma')\gamma'\rangle\\ &= -\int_0^L \langle X',Y'\rangle + \int_0^L\langle X,R(Y,\gamma')\gamma'\rangle,\end{aligned}$$

which is symmetric in X and Y.

Proof. First,

$$\begin{aligned}\frac{\partial}{\partial u}E(\gamma_{u.v}) &= \tfrac12\frac{\partial}{\partial u}\int_0^L\langle \tfrac{\partial}{\partial s}\alpha, \tfrac{\partial}{\partial s}\alpha\rangle\\ &= \int_0^L \langle \nabla_{\frac{\partial}{\partial u}}\tfrac{\partial}{\partial s}\alpha, \tfrac{\partial}{\partial s}\alpha\rangle\\ &= \int_0^L \langle \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial u}\alpha, \tfrac{\partial}{\partial s}\alpha\rangle\\ &= \langle \tfrac{\partial}{\partial u}\alpha, \tfrac{\partial}{\partial s}\alpha\rangle\Big|_0^L - \int_0^L\langle\tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\rangle\\ &= -\int_0^L\langle\tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\rangle.\end{aligned}$$

Now, let us take the second derivative:

$$
\begin{aligned}
&\frac{\partial^2}{\partial u \partial v} E(\gamma_{u.v}) \\
&\qquad = -\frac{\partial}{\partial v} \int_0^L \langle \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}} \tfrac{\partial}{\partial s}\alpha \rangle \\
&\qquad = -\int_0^L \langle \nabla_{\frac{\partial}{\partial v}} \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}} \tfrac{\partial}{\partial s}\alpha \rangle - \int_0^L \langle \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial s}} \tfrac{\partial}{\partial s}\alpha \rangle \\
&\qquad = -\int_0^L \langle \nabla_{\frac{\partial}{\partial v}} \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}} \tfrac{\partial}{\partial s}\alpha \rangle - \int_0^L \langle \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial s}} \tfrac{\partial}{\partial v}\alpha + R(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial s}) \tfrac{\partial}{\partial s}\alpha \rangle.
\end{aligned}
$$

Evaluation at $(u, v) = (0, 0)$ yields

$$
\frac{\partial^2}{\partial u \partial v} E(\gamma_{u,v})(0,0) = -\int_0^L \langle \nabla_{\frac{\partial}{\partial v}} \tfrac{\partial}{\partial u}\alpha \Big|_{(u,v)=(0,0)}, \gamma'' \rangle - \int_0^L \langle X, Y'' + R(Y, \gamma')\gamma' \rangle.
$$

With $\gamma'' = 0$ we obtain the desired result. □

10. Integration on manifolds

10.1 Orientability

Definition 10.1 (orientability)**.**

1. *An n-dimensional manifold M is called orientable if there exists* $\omega \in \Omega^n(M)$ *such that* $\omega_p \neq 0$ *for all* $p \in M$*.*
2. *An orientation on an n-dimensional manifold M is an equivalence class* $[\omega]$ *of nowhere vanishing n-forms*
$$\omega \sim \tilde{\omega} :\Leftrightarrow \tilde{\omega} = \lambda\omega$$
where $\lambda : M \to \mathbb{R}$ *is smooth and satisfies* $\lambda(p) > 0$ *for all* $p \in M$*.*
3. *Given an orentation* $[\omega]$ *on M, then we say a basis* $X_1, \ldots, X_n \in T_pM$ *is positively orientated if* $\omega(X_1, \ldots, X_n) > 0$*.*

Remark 10.2.
If M is connected and orientable, then there are exactly two orientations of M. Therefore, if M has k connected components and M is orientable, then M has 2^k orientations.

Example 10.3. If $\dim M = 2$ and $f : M \to \mathbb{R}^3$ is an immersion and $N : M \to \mathbb{R}^3$ is the unit normal, then for $X, Y \in T_pM$ define

$$\omega(X, Y) := \det\big(N_p,\ df(X),\ df(Y)\big)$$

Therefore M is orientable.

Definition 10.4 (orientation of charts)**.** *If M oriented, i.e. M comes with an orientation* $[\omega]$*, then a chart* $\varphi : U \to \mathbb{R}^n,\ \varphi = (x_1, \ldots, x_n)$ *is called positively oriented if*

$$\omega|_U = f\ dx_1 \wedge \cdots \wedge dx_n$$

with $f : U \to \mathbb{R},\ f(p) > 0$ *for all* $p \in U$*.*

Definition 10.5 (atlas)**.** *An atlas of M is a collection of charts* $\{U_\alpha\}_\alpha$ *such that* $M = \bigcup_\alpha U_\alpha$*.*

Theorem 10.6. *If M is oriented, then M has an atlas only consiting of positively oriented charts.*

Proof. Let $(U_\alpha, \varphi_\alpha)_{\alpha\in I}$ be an atlas of M, then without loss of generality we may assume that all U_α are connected.
We see that by definition

$$\omega|_{U_\alpha} = f_\alpha\ \varphi^*(dy_1 \wedge \cdots \wedge dy_n)$$

with either $f_\alpha(p) > 0$ or $f_\alpha(p) < 0$ for all $p \in U_\alpha$.
In the case of $f_\alpha < 0$ define $\tilde{\varphi}_\alpha : U \to \mathbb{R}$, $\tilde{\varphi}_\alpha = (-x_1, x_2, \dots, x_n)$ where $\varphi_\alpha = (x_1, \dots, x_n)$.
If we replace all negatively oriented φ_α by $\tilde{\varphi}_\alpha$, then these will do the trick. □

Remark 10.7. The converse of theorem 10.6 is also ture, but to proof this we will need thepartition of unity, so we will do it later on.

10.2 Integration of $\omega \in \Omega_0^n(M)$ with $\operatorname{supp}\omega \subset U$

Definition 10.8 (support of a section)**.** *If E is a vector bundle over M and $\psi \in \Gamma(E)$, then we define the support of ψ as*

$$\operatorname{supp}\psi := \overline{\{p \in M | \psi_p \neq 0\}}$$

Remark 10.9. The support is well defined as $\{p \in M | \psi_p \neq 0\}$, as the complement of the closed set $\{p \in M | \psi_p = 0\}$, is open.

Definition 10.10 ($\Omega_0^n(\mathbb{R}^n)$)**.** *We define $\Omega_0^n(\mathbb{R}^n) := \{\omega \in \Omega^n(\mathbb{R}^n) | \operatorname{supp}\omega$ is comapct$\}$.*

Definition 10.11 (Integral of $\omega \in \Omega_0^n(\mathbb{R}^n)$)**.** *If $\omega \in \Omega_0^n(\mathbb{R}^n)$, then we define*

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f$$

where we use the notation of definition 10.4.

Theorem 10.12. *Given $\omega, \tilde{\omega} \in \Omega_0^n(\mathbb{R}^n)$ and a diffeomorphism $\varphi : \operatorname{supp}\tilde{\omega} \to \operatorname{supp}\omega$ such that $\omega = \varphi^*\tilde{\omega}$ and $\det \varphi' > 0$, then*

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \tilde{\omega}$$

Proof. Let $\tilde{\omega} = \tilde{f}\ dx_1 \wedge \cdots \wedge dx_n$ and $\omega = f\ dx_1 \wedge \cdots \wedge dx_n$ with $f = \omega(X_1, \dots, X_n)$ where $X_k(p) = (p, e_k)$, $k = 1, \dots, n$. Then we see that

$$\begin{aligned}
[(\varphi^*\tilde{\omega})(X_1, \dots, X_n)]_p &= [\tilde{\omega}(d\varphi(X_1), \dots, d\varphi(X_n))]_p \\
&= \tilde{f} \circ \varphi(p) \det\Big(\varphi'_p(e_1) \cdots \varphi'_p(e_n)\Big) \\
&= \tilde{f} \circ \varphi(p) \det\Big(\varphi'_p\Big)
\end{aligned}$$

but also

$$[(\varphi^*\tilde{\omega})(X_1, \dots, X_n)]_p = [\omega(X_1, \dots, X_n)]_p = f(p)$$

for all $p \in \mathbb{R}^n$, i.e.

$$f = \tilde{f} \circ \varphi \det(\varphi') = \tilde{f} \circ \varphi |\det(\varphi')|$$

where we used $det(\varphi') > 0$ in the last equality. We finally yield, using the transormation of coordinates theorem

$$\int_{\mathbb{R}^n} \tilde{\omega} = \int_{\mathbb{R}^n} \tilde{f} = \int_{\mathbb{R}^n} \tilde{f} \circ \varphi |det(\varphi')| = \int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} \omega$$

□

Definition 10.13 (Integral of $\omega \in \Omega_0^n(M)$ with $\operatorname{supp} \omega \subset U$)**.** *If M is oriented, $\omega \in \Omega_0^n(M)$ and $\operatorname{supp} \omega \subset U$ where $U, \varphi)$is an orientation preserving chart with $V := \varphi(U)$, $\gamma : V \to M$, $\gamma = \varphi^{-1}$, then we define*

$$\int_M \omega := \int_{\mathbb{R}^n} \gamma^* \omega$$

Remark 10.14. The Integral is well defined, because assume $(\tilde{U}, \tilde{\varphi})$ is another chart such that $\tilde{\gamma} = \tilde{\varphi}^{-1}$ and the change of coordinates ψ, then $\tilde{\gamma}^*\omega = (\gamma \circ \psi)^*\omega = \psi^*\gamma^*\omega$. By theorem 10.12 and since ψ is orientation preserving, we see that

$$\int_{\mathbb{R}^n} \gamma^*\omega = \int_{\mathbb{R}^n} \tilde{\gamma}^*\omega$$

which makes the definition independent of the choice of (U, φ).

10.3 Partition of unity

Theorem 10.15 (partition of unity)**.** *Let M be a manifold, $A \subset M$ compact and $(U_\alpha)_{\alpha \in I}$ an open cover of A. Then there are $\varrho_1, \dots, \varrho_m \in C^\infty(M)$ such that for each $i \in 1, \dots, m$ there is $\alpha_i \in I$ such that $\operatorname{supp} \varrho_i \subset U_{\alpha_i}$ is compact. Moreover $\varrho_i(p) \geq 0$ for all $p \in M$ and $\sum_{i=0}^{m} \varrho_i(p) = 1$ for all $p \in A$.*

Proof. We already know that there is a function $g \in C^\infty(\mathbb{R}^n)$ such that $g(p) \geq 0$ for all $p \in \mathbb{R}^n$ and $g(p) > 0$ if $p \in D := \{x \in \mathbb{R}^n | |x| < 1\}$. Define

$$D_p := \{x \in \mathbb{R}^n | |x - p| < 1\}, \quad \tilde{D} := \{x \in \mathbb{R}^n | |x| < 2\}, \quad \tilde{D}_p := \{x \in \mathbb{R}^n | |x - p| < 2\}.$$

For each $p \in A$ there is a chart (U_p, φ_p) such that $\varphi_p : U_p \to \tilde{D}_p$ is a diffeomorphism and $U_p \subset U_\alpha$ for some $\alpha \in I$.
If we define

$$V_p := \varphi_p^{-1}(D_p)$$

then $(V_p)_{p \in A}$ is an open cover of A. This implies that there are $p_1, \dots, p_m \in A$ such that $A \subset V_{p_1} \cup \dots \cup V_{p_m}$.

Now define for $i \in \{1,\dots,m\}$ $\tilde{\varrho}_i : M \to \mathbb{R}$ by $\tilde{\varrho}_i = \begin{cases} 0 & \text{if } p \notin V_{p_i} \\ g \circ \varphi_{p_i} & \text{if } p \in V_{p_i} \end{cases}$
This means $(\tilde{\varrho_1} + \cdots + \tilde{\varrho_m})(p) \geq 0$ always holds, and in particular $(\tilde{\varrho_1} + \cdots + \tilde{\varrho_m})(p) > 0$ for $p \in A$. If now $A = M$, then we define

$$\varrho_i := \frac{\tilde{\varrho_i}}{(\tilde{\varrho_1} + \cdots + \tilde{\varrho_m})}$$

and we are done.
Otherwise, since A is compact, $\tilde{\varrho}_1 + \cdots + \tilde{\varrho}_m$ attains its minimum in A, i.e. there is an $\epsilon > 0$ such that $(\tilde{\varrho}_1 + \cdots + \tilde{\varrho}_m)(p) \geq \epsilon$ for all $p \in A$. Consturct $h \in C^\infty(M)$ such that $h(x) > 0$ for all $x \in \mathbb{R}$ and $h(x) = x$ for $x \geq \epsilon$. Now for $i \in \{1,\dots,m\}$ define

$$\varrho_i := \frac{\tilde{\varrho}_i}{h(\tilde{\varrho}_1 + \cdots + \tilde{\varrho}_m)}$$

then clearly $(\tilde{\varrho}_1 + \cdots + \tilde{\varrho}_m)|_A = 1$. □

Theorem 10.16 (partition of unity - general version). *Let M be a manifold and $(U_\alpha)_{\alpha \in I}$ an open cover of M, i.e. $\cup_{\alpha \in I} U_\alpha = M$. Then there is a family $(\varrho_\beta)_{\beta \in J}$ with $\varrho_\beta \in C^\infty(M)$ such that*

1. *for each $\beta \in J$ there is $\alpha \in I$ such that* $\operatorname{supp} \varrho_\beta \subset U_\alpha$ *is compact.*
2. *$\varrho_i(p) \geq 0$ for all $p \in M$.*
3. *$\varrho_\beta(p) \neq 0$ only for finitely many $\beta \in J$ and and $\sum_{\beta \in J} \varrho_\beta(p) = 1$ for all $p \in M$.*

Proof. This theorem will remain without proof, as we only cite it to emphasize the existence of a more general version, but do not actually use it much. □

Nonetheless we will shortly give some applications of the general partition of unity.

Theorem 10.17. *Every manifold has a Riemannian metric.*

Proof. We have coordinate charts $(U_\alpha, \varphi_\alpha)$, i.e. an open cover $(U_\alpha)_{\alpha \in I}$ of M and a Riemannian metric g_α on $U\alpha$, where $g_\alpha = \varphi_\alpha^* g_{\mathbb{R}^n}$ is the pullback metric. Now we choose a partition of unity $(\varrho_\beta)_{\beta \in J}$ subordinate to the cover $(U_\alpha)_{\alpha \in I}$.
Define

$$g := \sum_{\beta \in J} \varrho_\beta g_{\alpha(\beta)}$$

which is, as a linear combination of positive definite symmetric bilinear forms with positive coefficients, also positive definite. Then g is a Riemannian metric on M. □

Theorem 10.18. *Every vector bundle E over M has a connection.*

Remark 10.19. If $\nabla^1, \dots, \nabla^m$ are connections on any bundle E and $\lambda_1, \dots, \lambda_m \in C^\infty(M)$ with $\lambda_1 + \cdots + \lambda_m = 1$, then ∇ defined by

$$\nabla_X \psi := \lambda_1 \nabla^1_X \psi + \cdots + \lambda_m \nabla^m_X \psi$$

for $X \in \Gamma(TM)$, $\psi \in \Gamma(E)$, certainly is a connection.

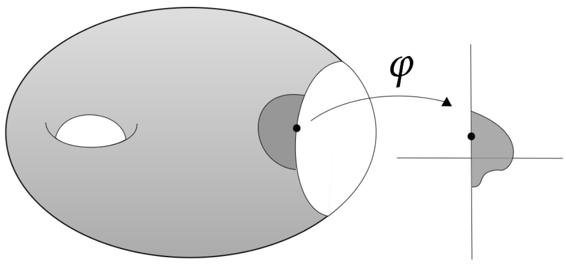
Proof. Locally (on U_α) E looks like $U_\alpha \times \mathbb{R}^k$, therefore, by pulling pack the trivial connection, we have a connection ∇^α on $E|_{U_\alpha}$. Choose a partition of unity $(\varrho_\beta)_{\beta\in J}$ subordinate to the cover $(U_\alpha)_{\alpha\in I}$, then we can say

$$\nabla = \sum_{\beta\in J} \varrho_\beta \nabla^{\alpha(\beta)}$$

is certainly a connection, as only finitely many of the $\varrho \neq 0$. □

10.4 Manifolds with a boundary

Definition 10.20 (manifold with a boundary)**.** *A n-dimensional manifold is said to have a boundary if its coodinate charts take values in $H := \{(x_1, \ldots, x_n) \in R^n \mid x_1 \geq 0\}$.*



Remark 10.21. If M is a manifold with boundary then

1. $M^\circ = M - \partial M$ is an n-dimensional manifold.
2. ∂M is an $(n-1)$-dimensional manifold without boundary.

Definition 10.22 (positive oriented)**.** *Let $p \in \partial M$ and $X_1, \ldots, X_{n-1} \in T_p\partial M$ then we say $X_1, \ldots, X_{n-1}$ are positive oriented if $(N, X_1, \ldots, X_{n-1})$ are positive oriented for some outward pointing $N \in T_p\partial M$.*

Definition 10.23 (outward pointing)**.** *A vector $N \in T_p\partial M$ is called outward pointing if $dx_1[(d\varphi)(N)] > 0$ for any coodinate chart φ.*

10.5 Integration of $\omega \in \Omega_0^n(M)$

We have already defined $\int_M \omega$ for $\omega \in \Omega_0^n(M)$ if $\operatorname{supp}\omega \subset U$, where (U, φ) is a chart. The main task now will be to get rid of the assumption that the support is contained in some chart neighbourhood. The strategy will be to cover $\operatorname{supp}\omega$ with U_α coming from charts and then to choose a subordinate partition of unity $(\varrho_\beta)_{\beta in J}$. As $\operatorname{supp}\omega$ is compact it is sufficient to choose a finite set $\{U_1, \ldots, U_m\}$ to cover it and therefore we only need a finite partition of unity $\varrho_1, \ldots, \varrho_m$.

Definition 10.24 (Integral over $\omega \in \Omega_0^n(M)$). *Let $\omega \in \Omega_0^n(M)$, $(U_\alpha, \varphi_\alpha)_{\alpha\in I}$ an open cover of M consisting of coodrinate charts and $\varrho_1, \dots, \varrho_m$ a partition of unity subordinate to $(U_\alpha)_{\alpha\in I}$ then we define*

$$\int_M \omega := \sum_{i=1}^m \int_M \varrho_i \omega$$

To ensure that our newly gained integral is well defined, we proof the following theorem.

Theorem 10.25. $\int_M \omega$ *thus defined is independet of the choices.*

Proof. Without loss of generality we can assume $(U_\alpha)_{\alpha\in I}$ contains all coordinate neighbourhoods, i.e. the independence of $(U_\alpha)_{\alpha\in I}$ is no problem at all.
For the independence of the partition of unity let $\varrho_1, \dots, \varrho_m, \tilde{\varrho_1}, \dots, \tilde{\varrho_m} \in C^\infty(M)$ be two partitions of unity, then

$$\sum_{i=1}^m \int_M \varrho_i \omega = \sum_{i=1}^m \int_M \left(\sum_{j=1}^m \tilde{\varrho}_j\right) \varrho_i \omega = \sum_{i,j=1}^m \int_M \tilde{\varrho}_j \varrho_i \omega = \sum_{j=1}^m \int_M \left(\sum_{i=1}^m \varrho_i\right) \tilde{\varrho}_j \omega = \sum_{j=1}^m \int_M \tilde{\varrho}_j \omega$$

□

Theorem 10.26 (Stoke's theorem). *Let M be an oriented manifold with boundary and $\omega \in \Omega_0^{n-1}(M)$, then*

$$\int_M d\omega = \int_{\partial M} \omega$$

Proof. As we habe the partition of unity as one of our tools, we may assume that without loss of generality $M = H^n = \{x \in \mathbb{R}^n | x_1 \leq 0\}$. For further simplification we also only consider the case $n = 2$ as the calculations work in the same manner for higher dimensions. We can write

$$\omega = a\mathrm{d}x + b\mathrm{d}y$$

then

$$\mathrm{d}\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y$$

By the definition of the integration of forms in $\mathbb{R}^n$ and Fubini's theorem we now see that

$$\begin{aligned}\int_{H^2} \mathrm{d}\omega &= \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial b}{\partial x}\, \mathrm{d}x\mathrm{d}y - \int_{-\infty}^{0} \int_{-\infty}^{\infty} \frac{\partial a}{\partial y}\, \mathrm{d}y\mathrm{d}x \\ &= \int_{-\infty}^{\infty} b(0, y)\, \mathrm{d}y - 0 \\ &= \int_{\partial H^2} b\, \mathrm{d}y \\ &= \int_{\partial H^2} \omega\end{aligned}$$

where we used the compact support of ω and the fact that the y-axis suits the induced orientation of ∂H^2. □

For now consider an oriented Riemannian manifold M, $p \in M$, an orthonormal basis $X_1, \ldots, X_n$ of T_pM and $\omega \in \Omega^0(M)$ that is compatible with the orientation and $\omega(X_1, \ldots, X_n) = 1$.
Then any other positive oriented basis $Y_1, \ldots, Y_n$ of T_pM is given by

$$Y_j = \sum_{i=1}^{n} a_{ij} X_i$$

with $A = (a_{ij}) \in SO(n)$, i.e. $A^T A = I$, $\det A = 1$. Also we can immediately see that $\omega(Y_1, \ldots, Y_n) = 1$.

Theorem 10.27. *On an oriented Riemannian manifold, there is a unique $\omega_M \in \Omega^n(M)$ such that*

$$\omega_M(X_1, \ldots, X_n) = 1$$

on each positive oriented orthonormal basis $X_1, \ldots, X_n \in TpM$.

Definition 10.28. *(volume form) The unique n-form ω_M as defined in theorem 10.27 is called the volume form of M.*

Definition 10.29. *(Integral of $f \in C_0^\infty(M)$) Let M be an oriented Riemannian manifold and $f \in C_0^\infty(M)$, then we define*

$$\int_M f := \int_M f \cdot \omega_M$$

11. Remarkable Theorems

11.1 Theorem of Gauß-Bonnet

We already know that for a compact Riemannian manifold M, and $f \in \mathscr{C}^\infty(\mathrm{M})$ one can define

$$\int_\mathrm{M} f \in \mathbb{R}$$

If M is any orientable manifold of dimension n and $\omega \in \Omega_0^n(\mathrm{M})$ then one can define

$$\int_\mathrm{M} \omega$$

In this chapter we will derive that there is an interesting relation between Topology and Geometry.

Theorem 11.1. *(Gauß-Bonnet) Let M be a compact and oriented Riemannian manifold with* $\dim M = 2$. *Then there is an integer $g \in 0, 1, 2, \ldots$ such that*

$$\int_M K = 4\pi(1-g)$$

Proof. We will later proof a more general version, the "Poincaré-Hopf Index"-theorem. □

Definition 11.2. *(genus) The integer g mentioned in theorem 11.1 is called the genus of M.*

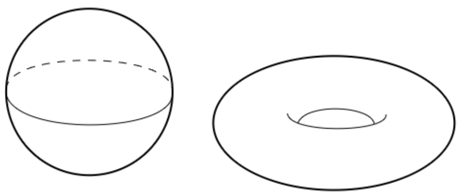


Figure 11.1: A sphere and a Torus that are of genus 0 and 1 respectively.

Remark 11.3. The genus of M does neither depend on the metric, nor on the orientation.

By considering some simple examples one can easily see that the Gauß-Bonnet theorem provides a remarkable link between a local property, namely the curvature, and a topological, thus global property of M. In general one could say "g is the number of holes in M". Another rather interesting theorem, although it is not so easy to proof (thus we won't do it) is the following

Theorem 11.4. *If $M, \tilde{M}$ are compact oriented surfaces with genus $g, \tilde{g}$ respectively, then M is diffeomorphic to $\tilde{M}$ if and only if $g = \tilde{g}$*

So by listing the genus of a surface, we obtain complete list of diffeomorphism types of compact oriented surfaces.
One may know the follwing theorem from lectures about complex analysis which enables us to classify also the curvature of a surface by in dependence of its genus.

Theorem 11.5. *(uniformization theorem) Let M be a compact Riemannian surface, then there is $u \in C^\infty(M)$ such that regarding*

$$\langle \cdot, \cdot \rangle^\sim = e^{2u} \langle \cdot, \cdot \rangle$$

M has constant curvature $K = \{-1, 0, 1\}$ (depending on whether $g = 0, g = 1, g \geq 2$). For $g \geq 2$ u is unique.

Proof. This theorem is known from lectures about complex analysis. ☐

Theorem 11.6. *(Ricci-flow) On a compact surface the Ricci-flow*

$$\langle \cdot, \cdot \rangle^\bullet = -\mathrm{Ric} + \int_M K \langle \cdot, \cdot \rangle$$

converges to a conformally equivalent metric with constant curvature.

Definition 11.7. *(isometric immersion) Consider a Riemannian manifold M with $\langle \cdot, \cdot \rangle$ and $f : M \to \mathbb{R}^3$ then f is an isometric immersion if*

$$\langle df(X), df(Y) \rangle_{\mathbb{R}^3} = \langle X, Y \rangle_M$$

Theorem 11.8. *If M is compact and $f : M \to \mathbb{R}^3$ is an isometric immersion then there is $p \in M$ with $K(p) > 0$.*

Remark 11.9. This explains why we have to use the various models, that are well known from lectures about geometry, to visualize the hyperbolic plane . We cannot find an isometric immersion to $\mathbb{R}^3$ as this would be in contradiction to the constant negative curvature of hyperbolic space.

11.2 Bonnet-Myers's Theorem

Definition 11.10 (Simply connected)**.** *A manifold* M *is called simply connected if for every smooth map* $\gamma \colon \mathbb{S}^1 \to \mathrm{M}$, $\mathbb{S}^1 = \partial D^2$, *there is a smooth map* $f \colon D^2 \to \mathrm{M}$ *such that* $\gamma = f|_{\mathbb{S}^1}$.

Theorem 11.11. *Let* M *be a simply connected complete Riemannian manifold with constant sectional curvature* $K > 0$. *Then* M *is isometric to a round sphere of radius* $r = 1/\sqrt{K}$.

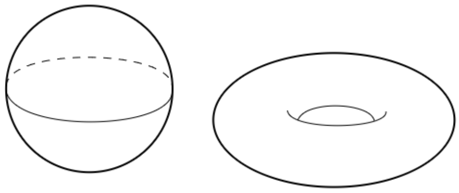


Figure 11.2: A sphere is simply connected and a Torus is not.

Proof. Without proof. □

Without completeness:
Then it is only a part of the sphere.

Without simply connected:
$\mathbb{RP}^n$ has also constant sectional curvature. Similar with lense spaces: Identify points on $\mathbb{S}^3 \subset \mathbb{C}^2$ that differ by $e^{2\pi i/n}$, $\mathrm{M} = \mathbb{S}^3/\sim$.

Theorem 11.12. *Let* M *be a simply connected, complete manifold, and let for all sectional curvatures* K_E *the inequality* $\frac{1}{4} < K_E \leq 1$ *hold, then* M *is homeomorphic to* $\mathbb{S}^n$*.*

Proof. As we want to save some time we will skip this quite complicated proof. □

Remark 11.13. For $\mathrm{M} = \mathbb{CP}^n$ one has $\frac{1}{4} \leq K_E \leq 1$.

Definition 11.14 (Scalar curvature)**.** *Let* M *Riemannian manifold,* $p \in \mathrm{M}$*,* $G_2(\mathrm{T}_p\mathrm{M})$ *Grassmanian of* 2*-planes* $E \subset \mathrm{T}_p\mathrm{M}$ *(* $\rightsquigarrow \dim G_2(\mathrm{T}_p\mathrm{M}) = n(n-1)/2$*), then*

$$\tilde{S} := \frac{1}{\mathrm{vol}(G_2(\mathrm{T}_p\mathrm{M}))} \int_{G_2(\mathrm{T}_p\mathrm{M})} K_E$$

is called the scalar curvature.

Definition 11.15 (Ricci curvature)**.** *Let* M *be a Riemannian manifold,* $p \in \mathrm{M}$ *and* $X \in \mathrm{T}_p\mathrm{M}$ *woth* $|X| = 1$.
Let further $\mathbb{S}^{n-2} \subset X^\perp \subset \mathrm{T}_p\mathrm{M}$*, then*

$$\widetilde{\mathrm{Ric}}(X,X) = \frac{1}{\mathrm{vol}(\mathbb{S}^{n-2})} \int_{\mathbb{S}^{n-2}} K_{\mathrm{span}\{X,Y\}} dY$$

is called Ricci curvature.

Let us try something simpler:
Ricci-tensor: Choose an orthonormal basis $Z_1, \ldots, Z_n$ of $\mathrm{T}_p\mathrm{M}$ with $Z_1 = X$ and define

$$\mathrm{Ric}(X,X) := \frac{1}{n-1} \sum_{i=1}^{n} \langle R(Z_i, X)X, Z_i \rangle = \frac{1}{n-1} \sum_{i=2}^{n} K_{\mathrm{span}\{X,Z_i\}}.$$

Then with $AZ := R(Z,X)X$ defines an endomorphism of $\mathrm{T}_p\mathrm{M}$ and

$$\mathrm{Ric}(X,X) := \frac{1}{n-1} \sum_{i=1}^{n} \langle R(Z_i, X)X, Z_i \rangle = \frac{1}{n-1} \sum_{i=1}^{n} \langle AZ_i, Z_i \rangle = \frac{1}{n-1} \mathrm{tr}(A).$$

Thus $\mathrm{Ric}(X,X)$ does not depend on the choice of the basis.

Definition 11.16 (Ricci-tensor). *The bilinear map*

$$\mathrm{Ric}_p : T_pM \times T_pM \to \mathbb{R},\ (X,Y) \mapsto \mathrm{Ric}(X,Y)_p := \tfrac{1}{n-1}\mathrm{tr}(Z \mapsto R(Z,X)Y)$$

is called the Ricci-tensor of M at p.

Theorem 11.17. $\mathrm{Ric}_p\colon \mathrm{T}_p\mathrm{M} \times \mathrm{T}_p\mathrm{M} \to \mathbb{R}$ *is symmetric.*

Proof.

$$\mathrm{Ric}(X,Y) = \tfrac{1}{n-1}\sum_{i=1}^{n}\langle R(Z_i,X)Y, Z_i\rangle = \tfrac{1}{n-1}\sum_{i=1}^{n}\langle R(Z_i,Y)X, Z_i\rangle = \mathrm{Ric}(Y,X).$$

□

Now we have two symmetric bilinear forms on each tangent space, namely $\langle .,.\rangle$ and Ric.

Theorem 11.18. *The Ricci-tensor is well defined in the sense that it yields the Ricci-curvature, i.e.*

$$\widetilde{\mathrm{Ric}}(X,X) = \mathrm{Ric}(X,X)$$

Proof. Without proof. □

Definition 11.19 (ric_p-map). *Define*

$$\mathrm{ric}_p : \mathrm{T}_pM \to \mathrm{T}_pM \quad by \quad \langle \mathrm{ric}_p X, Y\rangle := \mathrm{Ric}(X,Y)$$

Remark 11.20. ric_p is self-adjoint.

$\rightsquigarrow$ Then the Eigenvalues $\kappa_1,\ldots,\kappa_n$ of ric_p (and eigenvectors) provide useful information which leads us to the following definition.

Definition 11.21 (Scalar curvature). *Let $Z_1,\ldots,Z_n$ be an orthonormal basis of* $\mathrm{T}_p\mathrm{M}$. *Then define*

$$S(p) := \tfrac{2}{n(n-1)}\sum_{i<j}\langle R(Z_i,Z_j)Z_j, Z_i\rangle$$

Remark 11.22. If we choose an ONB, then we can derive that

$$\begin{aligned}
\tfrac{1}{n}\sum_{j=1}^{n}\mathrm{Ric}(Z_j,Z_j) &= \tfrac{1}{n}\sum_{j=1}^{n}\tfrac{1}{n-1}\sum_{i\neq j}\langle R(Z_i,Z_j)Z_j, Z_i\rangle \\
&= \tfrac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\langle R(Z_i,Z_j)Z_j, Z_i\rangle \\
&= \tfrac{2}{n(n-1)}\sum_{i<j}\langle R(Z_i,Z_j)Z_j, Z_i\rangle \\
&= \tfrac{1}{n}\sum_{j=1}^{n}\langle \mathrm{ric}_p Z_j, Z_j\rangle \\
&= \tfrac{1}{n}\mathrm{tr}(\mathrm{ric}_p)
\end{aligned}$$

In other words, $\frac{1}{n}\mathrm{tr}(\mathrm{ric}_p)$ would also be a suitable definition for the scalar curvature.

Theorem 11.23. *The scalar curvature is well defined, i.e.* $\tilde{S}(p) = S(p)$.

Proof. Without proof. □

If we now restrict ourselves to the case $\dim M = 2$, then we can consider $K \in C^\infty(M)$, the sectional curvature of M. For $|X| = 1$, then $\mathrm{Ric}(X, X)$ is the average of all sectional curvatures of planes $E \subset T_pM$ with $X \subset E$. As $\dim M = 2$, it is $\mathrm{Ric}(X, X) = K$ for all $|X| = 1$, or even more general

$$\mathrm{Ric}(X, X) = K \langle X, X\rangle$$

for all $X \in T_pM$. So we can consider $\mathrm{Ric} = K \langle \cdot, \cdot\rangle$ as a map

$$\mathrm{Ric} : T_pM \times T_pM \to \mathbb{R}, \ X \mapsto K \langle X, X\rangle$$

This map turns a Riemannian manifold into an "Einstein"-manifold.

Definition 11.24 (Diameter)**.** *Let* M *be a Riemannian manifold, then*

$$\mathrm{diam}(\mathrm{M}) := \sup\{d(p, q) \mid p, q \in \mathrm{M}\} \in \mathbb{R} \cup \{\infty\}$$

is called the diameter of M.

Theorem 11.25. *Let* M *be a complete manifold, then*

$$\mathrm{diam}(\mathrm{M}) < \infty \Leftrightarrow M \text{ is compact}$$

Proof.

"⇒": $\mathrm{diam}(\mathrm{M}) < \infty$, then M closed and bounded, thus compact.

"⇐": $d\colon \mathrm{M} \times \mathrm{M} \to \mathbb{R}$ is continuous, thus takes its maximum. $\rightsquigarrow \mathrm{diam}(\mathrm{M}) < \infty$.

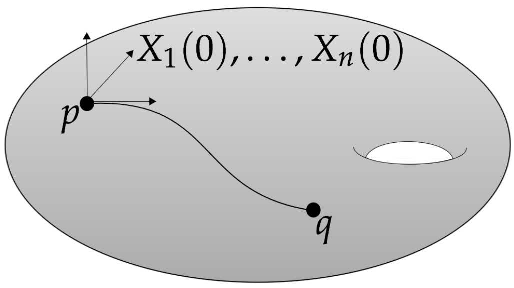
□

Theorem 11.26 (Bonnet-Myers)**.** *Let* M *be a complete Riemannian manifold such that*

$$\mathrm{Ric}(X, X) \geq \frac{1}{r^2}\langle X, X\rangle$$

holds for all $X \in \mathrm{TM}$*, then*

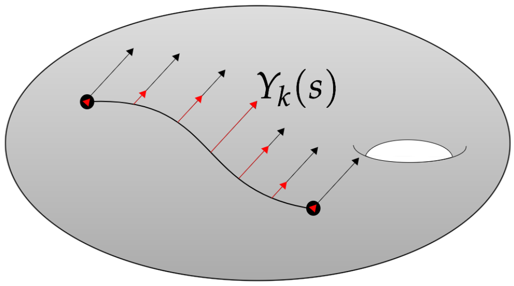
$$\mathrm{diam}(\mathrm{M}) \leq \pi r$$



Proof. Choose $p, q \in M$. $L := d(p,q) > 0$.
By Hopf-Rinow there is an arclength-parametrized geodesic $\gamma\colon [0, L] \to M$ with $\gamma(0) = p$ and $\gamma(L) = q$, i.e. $L = d(p,q)$.

Now choose a parallel orthonormal frame field $X_1, \ldots, X_n$ along γ with $X_1 = \gamma'$ and define vector fields $Y_i \in \Gamma(\gamma^* TM)$ by

$$Y_i(s) = \sin\left(\tfrac{\pi s}{L}\right) X_i(s)$$



We know want to create a variation with fixed end points, therefore define variations

$$\tilde{\alpha}_i\colon (-\varepsilon, \varepsilon) \times [0, L] \to M$$

of γ by

$$\tilde{\alpha}_i(t,s) = \exp(tY_i(s)) \text{ and denote } \gamma_t^i = \tilde{\alpha}_i(t,.)$$

Thus with

$$\alpha_i : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times [0, L] \to M,\ \alpha_i(u,v,s) := \tilde{\alpha}_i\left(\tfrac{u+v}{2}, s\right)$$

we have

$$X(s) := \left.\frac{\partial \alpha_i}{\partial u}\right|_{(0,0,s)} = Y_i(s)$$
$$Y(s) = \left.\frac{\partial \alpha_i}{\partial v}\right|_{(0,0,s)} = Y_i(s)$$

Then we use the second variational formula of length: If $g\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$, $g(t) = L(\gamma_t)$, then g has a global minimum at $t = 0$, i.e. $0 \le g''(0)$. Thus with $\eta_t(s) := \tilde{\alpha}(t,s) = \alpha(t,t,s)$

$$0 \le g''(0) = \tfrac{d^2}{dt^2}|_{t=0} E(\eta_t) = \tfrac{\partial^2}{\partial u \partial v}|_{u=v=0} L(\gamma^i_{u,v}) = -\int_0^L \langle Y_k, Y_k'' + R(Y_k, \gamma')\gamma' \rangle$$

Since $Y_k(s) = \sin(\frac{\pi s}{L}) X_k$, we have $Y_k''(s) = -(\frac{\pi}{L})^2 \sin(\frac{\pi s}{L}) X_k(s)$. Thus, for each k,

$$\left(\tfrac{\pi s}{L}\right)^2 \int_0^L \sin^2\left(\tfrac{\pi}{L}\right) = -\int_0^L \langle Y_k'', Y_k \rangle \ge \int_0^L \langle R(Y_k, \gamma')\gamma', Y_k \rangle = \int_0^L \sin^2\left(\tfrac{\pi s}{L}\right) \langle R(X_k, \gamma')\gamma', X_k \rangle.$$

By assumption $\mathrm{Ric}(X,X) \geq \frac{1}{r^2}\langle X,X\rangle$. Thus summing over $k = 2,\dots,n$ we get

$$\begin{aligned}\tfrac{n-1}{r^2}\int_0^L \sin^2(\tfrac{\pi s}{L}) &\leq (n-1)\int_0^L \sin^2(\tfrac{\pi s}{L})\mathrm{Ric}(\gamma',\gamma') \\ &= \sum_{k=2}^{n}\int_0^L \sin^2(\tfrac{\pi s}{L})\langle R(X_k,\gamma')\gamma', X_k\rangle \\ &\leq (n-1)(\tfrac{\pi}{L})^2\int_0^L \sin^2(\tfrac{\pi s}{L}).\end{aligned}$$

As $\int_0^L \sin^2(\frac{\pi s}{L}) > 0$, this means that $\frac{\pi^2}{L^2} \geq \frac{1}{r^2}$ and since $r \geq 0$ and $(\cdot)^2$ is monotone we have

$$\frac{\pi}{L} \geq \frac{1}{r} \Leftrightarrow L \leq \pi r$$

Then, since $L = d(p,q)$ we have $d(p,q) \leq \pi r$ so as p and q are arbitrary we finally yield $\mathrm{diam}(M) \leq \pi r$. □

11.3 Poincaré-Hopf Index Theorem

For now the setup will be the following: Let M be an oriented manifold of dimension 2 not necessarily with a boundary and E an oriented euclidean rank 2 vector bundle over M.

Remark 11.27. If V is some 2-dimensional vector space with det defined on it, then we get a 90°-rotation

$$J : V \to V \;\text{ with }\; \det(X,Y) = \langle X, JY\rangle \;\text{ and }\; J^2 = I$$

This means, that if $\lambda = \alpha + i\beta \in \mathbb{C}$ and $\psi \in V$ then

$$\lambda \cdot V := \alpha\psi + \beta J\psi$$

This means V becomes a 1-dimensional complex vector space. In this sense the vector bundle E can also be viewed as a 1-dimensional vector bundle, a "complex-line-bundle".

Now suppose E has $\psi \in \Gamma(E)$ with $\psi_p \neq 0$ for all $p \in M$, i.e. a nowhere vanishing section. Then E also has a section $\psi \in \Gamma(E)$ that satisfies $|\psi_p| = 1$ for all $p \in M$. Now, as we already know, E comes wirh a metric connection ∇.
For some $X \in T_pM$ we have

$$0 = \tfrac{1}{2}X\,\langle\psi,\psi\rangle = \langle\nabla_X\psi,\psi\rangle$$

so

$$\nabla\psi = \eta J\psi \;\text{ with }\; \eta \in \Omega^1(M)$$

Definition 11.28. *(rotation form) The 1-form η in our consideration above is called the rotation 1-form of ψ.*

Theorem 11.29. *Regarding our considerations above it holds that*

$$R^{\nabla}(X,Y)\psi = \mathrm{d}\eta(X,Y)J\psi$$

Proof. Let $X, Y \in T_pM$ and $\psi \in \Gamma(E)$ then with

$$\nabla_Y\psi = \eta(Y)J\psi \quad \nabla_X\psi = \eta(X)J\psi \quad \nabla_{[X,Y]}\psi = \eta([X,Y])J\psi$$

we can easily calculate

$$\begin{aligned} R^{\nabla}(X,Y)\psi &= \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi \\ &= (X\eta(Y)J\psi - Y\eta(X)J\psi - \eta([X,Y]))\,J\psi + \eta(Y)J\nabla_X\psi - \eta(X)J\nabla_Y\psi \\ &= (\mathrm{d}\eta(X,Y) - \eta([X,Y]))\,J\psi + \eta(Y)J\eta(X)J\psi - \eta(X)J\eta(Y)J\psi \\ &= \mathrm{d}\eta(X,Y)J\psi - \eta([X,Y])J\psi - \eta(Y)\eta(X)\psi + \eta(X)\eta(Y)\psi \\ &= \mathrm{d}\eta(X,Y)J\psi \end{aligned}$$

where we used that $JJ = -I$. □

Example 11.30. Let M be a Riemannian manifold, $E = TM$ and $Y \in \Gamma(TM)$ with $|Y| = 1$. Then with $X := -JY$ we have for the volume form

$$\omega_M(X,Y) = 1$$

as well as

$$K = \langle R(X,Y)Y, X\rangle = \langle \mathrm{d}\eta(X,Y)JY, X\rangle = -\mathrm{d}\eta(X,Y)$$

or in other words, we yield that

$$K\omega_M = -\mathrm{d}\eta$$

Definition 11.31 (curvature form)**.** *The 2-form $\Omega \in \Omega^2(M)$ defined by $R^{\nabla} = -\Omega J$ is called the curvature 2-form of ∇.*

Theorem 11.32. *If E has a nowhere vanishing section and $\partial M = \emptyset$, then the curvature form Ω of any metric connection on E satisfies*

$$\int_M \Omega = 0$$

Proof. We have that $\Omega = -\mathrm{d}\eta$, therefore Stoke's theorem tells us that

$$\int_M \Omega = -\int_{\partial M} \mathrm{d}\eta = -\int_{\emptyset} \mathrm{d}\eta = 0$$

□

Corollary 11.33. *For any Riemannian metric on $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ we have $\int_{T^2} K = 0$.*

Corollary 11.34 (Hairy ball theorem)**.** *There is no vector field $X \in \Gamma(T\mathbb{S}^2)$ without zeros.*

A little Differential Topology

Goal: Every rank n vector bundle over an n-dimensional manifold has a section ψ with only isolated zeros.
To achieve this goal we will need a short introduction into differential topology, but as this is no course on this topic we will only clarify some vocabulary.

Definition 11.35 (convergence in C^∞-topology)**.** *A sequence of maps $f_1, f_2, \ldots : M \to \tilde{M}$ is said to converge in the C^∞-topology, if their representations in charts converge uniformly on compact subsets, together with all their partial derivatives.*

Definition 11.36 (transversal intersection)**.** *Two maps $f_i : M_i \to \tilde{M}$ $i = 1, 2$ are said to intersect transversally if for every $p_1 \in M_1$, $p_2 \in M_2$ with $q := f_1(p_1) = f_2(p_2)$ we have that*

$$df_1(T_{p_1}M_1) + df_2(T_{p_2}M_2) = T_q\tilde{M}$$

Theorem 11.37 (transversality theorem)**.** *Given a smooth map $f_1 : M_1 \to \tilde{M}$, then the set of those smooth $f_2 : M_2 \to \tilde{M}$ which are transversal to f_1 is dense in the C^∞-topology.*

Example 11.38. Given two transversal smooth maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}^3$, then

$$f_1(\mathbb{R}) \bigcap f_2(\mathbb{R}) = \emptyset.$$

Now, let $E \to M$ be a rank n vector bundle over an n-dimensional manifold and $\rho_0 \in \Gamma(E)$ *the zero section*, i.e. $\rho_0(p) = 0 \quad \forall p \in M$. Then the transversality theorem yields that there is $\psi \in \Gamma(E)$ intersecting ρ_0 transversally. Let $p \in M$ be a zero of ψ. Then, by using a frame field and coordinates on M, we can assume without loss of generality that $\psi \colon \mathbb{R}^n \to \mathbb{R}^{2n}$ is the graph of a function f transversal to the graph of the zero function. In particular, the differential of f must have full rank, which implies that ψ has an isolated zero.

If M is a compact oriented surface and E is an oriented Euclidean vector bundle over M, i.e. a hermitian line bundle, then these zeros come with sign—the determinant of the differential of f—which is the *winding number* $ind_p\psi \in \{-1, 1\}$ *of* ψ *around* p (running around a small positively oriented circle around p).
If $J \in \Gamma\mathrm{End}(E)$ denotes the positive 90-degree rotation the we have seen that the curvature of a connection ∇ on E satisfies

$$R^\nabla = -\Omega^\nabla J$$

for some $\Omega^\nabla \in \Omega^2 M$.

Theorem 11.39. *Let $E \to M$ be a hermtian line bundle with connection ∇. Then*

$$\deg E := \tfrac{1}{2\pi} \int_M \Omega^\nabla$$

is independent of ∇.

Proof. is left as an exercise. ☐

Theorem 11.40 (Poincaré–Hopf Index Theorem). *Let $E \to M$ be a hermtian line bundle over a compact oriented surface and $\psi \in \Gamma E$ have only isolated zeros $p_1, \dots, p_n \in M$, then*

$$\deg E = \sum_i \mathrm{ind}_{p_i} \psi .$$

Proof. Choose a metric connection ∇ on E and charts (U_i, φ_i) with $U_i \ni p_i$, $\varphi_i(U_i) = B_2(0)$ and $\varphi_i(p_i) = 0$. Let $V_i = \varphi_i^{-1}(B_\varepsilon(0))$. Then $M_\varepsilon := M \setminus \cup_i \mathring{V}_i$ is a manifold with boundary,

$$\deg E = \lim_{\varepsilon \to 0} \int_{M_\varepsilon} \Omega^\nabla .$$

Without loss of generality we can assume that $|\psi| = 1$ on M_ε. Then $\Omega^\nabla = -d\eta$, where η denotes the rotation form of ψ. Thus

$$\int_{M_\varepsilon} \Omega^\nabla = -\int_{\partial M_\varepsilon} \eta = \sum_i \int_{\partial V_i} \eta .$$

Now choose $\varphi_i \in \Gamma(E|_{U_i})$ with $|\varphi_i| = 1$. Then $\psi|_{U_i} = y_1 \varphi_i + y_2 J \varphi_i$ and thus

$$\begin{aligned} \int_{\partial V_i} \eta &= \int_{\partial V_i} \langle dy_1 \varphi_i + y_1 \nabla \varphi_i + dy_2 \varphi_i + y_2 \nabla \varphi_i, y_1 J \varphi_i - y_2 \varphi_i \rangle \\ &= \int_{\partial V_i} y_1 dy_2 - y_2 dy_1 + \underbrace{(y_1^2 + y_2^2)}_{=1} \langle \nabla \varphi_i, J \varphi_i \rangle \\ &= 2\pi \, \mathrm{ind}_{p_i} \psi - \int_{V_i} \Omega^\nabla \end{aligned}$$

□

11.4 Gauß–Bonnet Theorem

Poincaré–Hopf index theorem: Let M be a compact oriented surface and $E \to M$ be a rank 2 oriented Euclidean vector bundle over M. Then there is $\deg E \in \mathbb{Z}$ such that

1. For every metric connection ∇ on E we have $R^\nabla = -\Omega^\nabla J$ with $\Omega^\nabla \in \Omega^2(M; \mathbb{R})$ and

$$\int_M \Omega^\nabla = 2\pi \deg E .$$

2. For every $\psi \in \Gamma E$ with isolated zeros $p_1, \dots, p_n$, then

$$\sum_i \mathrm{ind}_{p_i} \psi = \deg E .$$

Recall: If $\langle ., . \rangle$ is a Riemannian metric on an oriented surface with area form $d\mathrm{vol}_M$, then

$$\Omega^\nabla = K \, d\mathrm{vol}_M .$$

Corollary 11.41. *If M is a compact oriented Riemannian surface with sectional curvature K, then*

$$\int_M K \, d\mathrm{vol}_M = 2\pi \deg TM .$$

Corollary 11.42. *The Euler characteristic* $\chi(M) := \deg TM$ *does not depend on the Riemannian metric.*

Proof. Given two Riemannian metrics $\langle.,.\rangle_i$ on M. Then linear interpolation yields a continuous family of metrics $\langle.,.\rangle_t$, $t \in [0,1]$. Also K and $d\mathrm{vol}_M$ depend continuously on t, and thus the degree of TM which as integer number must then be constant. □

Definition 11.43. *Let M be a manifold, $f \in C^\infty(M)$. Then $p \in M$ is called a critical point, if $d_p f = 0$.*

Exercise 11.44. If $p \in M$ is a critical point and $\varphi = (x_1, \ldots, x_m)$ is a chart around p and we define $\mathrm{Hess}_p \colon T_pM \times T_pM \to \mathbb{R}$ by

$$\mathrm{Hess}_p(X,Y) = \sum_{i,j} \tfrac{\partial^2 f}{\partial x_i \partial x_j} a_i b_j \,,$$

where $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x_j}$. In fact, if γ is a curve such that $\gamma(0) = p$, then

$$\mathrm{Hess}_p(\gamma',\gamma') = \tfrac{d^2}{dt^2}(f \circ \gamma)\,.$$

If V is a finite-dimensional Euclidean vector space, then:

a) For $X \in V$ define $X^\flat \int V^*$ by $X^\flat(Y) = \langle X, Y\rangle$.

b) For $\omega \in V^*$ define $\omega^\sharp \int V$ by $\omega(Y) = \langle \omega^\sharp, Y\rangle$.

Definition 11.45. *If M is Riemannian and $f \in C^\infty M$. Then the gradient of f is defined by* $\mathrm{grad} f := (df)^\sharp \in \Gamma(TM)$*, i.e.*

$$\langle \mathrm{grad} f, X\rangle = df(X)\,.$$

Definition 11.46. *Let M be a Riemannian manifold, $p \in M$ and $f \in C^\infty M$. Then the Hessian of f is defined as*

$$\mathrm{Hess}_p f(X,Y) := \langle \nabla_X \mathrm{grad} f, Y\rangle\,.$$

Exercise 11.47. Show that the two notions of Hessians are consistent when we are at a critical point.

Theorem 11.48. *The Hessian is symmetric:*

$$\mathrm{Hess}_p(X,Y) = \mathrm{Hess}_p(Y,X)\,.$$

Proof.

$$\langle \nabla_X \mathrm{grad} f, Y\rangle = X\langle \mathrm{grad} f, Y\rangle - \langle \mathrm{grad} f, \nabla_X Y\rangle - YXf + \langle \mathrm{grad} f, \nabla_Y X\rangle$$
$$\langle \nabla_Y \mathrm{grad} f, X\rangle = [X,Y]f - \langle \mathrm{grad} f, \nabla_X Y - \nabla_Y X\rangle\,.$$

□

Definition 11.49. *Let M be a manifold, $f \in C^\infty M$. Then:*

a) *A critical point $p \in M$ is called Morse-critical point if $\operatorname{Hess}_p f$ is non-degenerate.*

b) *f is called a Morse function, if all critical points of f are Morse. ($\rightsquigarrow$ critical points are isolated)*

The following theorem is a variant of the transversality theorem.

Theorem 11.50. *Morse functions are dense in $C^\infty M$ with respect to the C^∞-topology.*

Morse-critical points on a surface are either

$$\begin{cases} \text{local minima, i.e. } \operatorname{Hess}_p f \text{ is positive-definite,} \\ \text{local maxima, i.e. } \operatorname{Hess}_p f \text{ is negative-definite,} \\ \text{saddle point, i.e. } \operatorname{Hess}_p f \text{ is indefinite.} \end{cases}$$

Theorem 11.51. *Let M be a compact oriented surface and $f \in C^\infty M$ be a Morse function. Then*

$$\chi(M) = \#\,minima - \#\,saddles + \#\,maxima\,.$$

Proof. Immediately follows from the Poincaré–Hopf index theorem. □

Remark 11.52. If we have a cell-decomposition of the surface, then we can construct a Morse function which has exactly one local maximum on each face, exactly one saddle on each edge and a local minimum at each vertex. Thus

$$\chi(M) = \#\,vertices - \#\,edges + \#\,faces\,.$$

Theorem 11.53 (Classification of surfaces)**.** *Two connected compact oriented surfaces are diffeomorphic if and and only if they have the same Euler characteristic.*

Theorem 11.54 (Poincaré-Hopf)**.** *Let M be a compact, oriented surface without boundary, E an oriented euclidean* rank 2 *vector bundle over M with metric connection ∇ and $\psi \in \Gamma(E)$ with isolated zeros $p_1, \dots, p_k$, then*

$$\int_M \Omega^\nabla = 2\pi \sum_{j=1}^k ind_{p_j}\psi$$

Proof. As our semester is already over we will not proof this remarkable theorem, but rather enjoy its beauty. □

Remark 11.55. One can also define the notion of the degree of a vector bundle by

$$\deg E := \sum_{j=1}^k ind_{p_j}\psi$$

which simplifies the above equation from theorem 11.54 to

$$\int_M \Omega^\nabla = 2\pi \deg E$$

In particular for oriented surfaces the equality $\deg(TM) = \chi(M) = 2(1-g)$ holds, where $\chi(M)$ is the "Euler characteristic" and g the genus of M. So in this special case we have

$$\chi(M) \begin{cases} \leq 2 \\ \in 2\mathbb{Z} \end{cases}$$

which usually would have to be checked, but for lack of time we just skip this.
As a final interesting observation we see that

$$\chi(\mathbb{S}^2) = \deg(T\mathbb{S}^2) = 2 \rightsquigarrow \int_{\mathbb{S}^2} K = 4\pi$$

11.5 Analysis on Riemannian manifolds

Let M be an n-dimensional oriented manifold. For $f \in C^\infty M$ we have $\operatorname{grad} f \in \Gamma TM$ defined by $\langle \operatorname{grad} f, X\rangle = df(X) = Xf$. Using the sharp operator this means $(df)^\sharp = \operatorname{grad} f$. Equivalently, $df = (\operatorname{grad} f)^\flat$.
On each tangent space T_pM there is a unique volume form $(d\mathrm{vol}_M)_p$ such that

$$d\mathrm{vol}_M(X_1, \ldots, X_n) = 1$$

for each positively oriented orthonormal basis $X_1, \ldots, X_n$ of T_pM. This form $d\mathrm{vol}_M \in \Omega^2 M$. Given $Y \in \Gamma TM$ we can build the interior product $i_Y d\mathrm{vol}_M \in \Omega^{n-1}M$ which is given by inserting Y into the first slot of $d\mathrm{vol}_M$,

$$i_Y d\mathrm{vol}_M(X_2, \ldots, X_n) = d\mathrm{vol}_M(Y, X_2, \ldots, X_n)\,.$$

Definition 11.56. *Let $Y \in \Gamma TM$. The divergence* $\operatorname{div} Y \in C^\infty M$ *of Y is then defined by*

$$\operatorname{div} Y \, d\mathrm{vol}_M = d(i_Y d\mathrm{vol}_M)\,.$$

Theorem 11.57. $\operatorname{div} Y = \operatorname{tr} \nabla Y$.

Proof. Let $X_1, \ldots, X_n \in \Gamma TM$ be an orthonormal local frame field which is parallel along radial geodesics outgoing form p. In particular, ∇X_i vanishes at p. Let $\omega_i \in \Omega^1 M$ denote the corresponding dual frame field, i.e. $\omega_i(X_j) = \delta_{ij}$. We have

$$\begin{aligned} d\omega_i(X_j, X_k)_p &= X_{j,p}\omega_i(X_k) - X_{k,p}\omega_i(X_j) - \omega_i([X_j, X_k]_p) \\ &= X_{j,p}\delta_{ik} - X_{k,p}\delta_{ij} - \omega_i((\nabla_{X_j} X_k - \nabla_{X_k} X_j)_p) = 0\,. \end{aligned}$$

Now, let $\sum y_i X_i := Y$ and define

$$\eta := \sum_{i=1}^{n} (-1)^{i-1} y_i \omega_1 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \omega_n\,.$$

Inserting ordered $(n-1)$-tuples of X_j's one easily verifies that $\eta = i_Y d\mathrm{vol}_M$. Moreover,

$$(d(i_Y d\mathrm{vol}_M))_p = (\sum_{i=1}^{n} \omega_1 \wedge \cdots \wedge dy_i \wedge \cdots \wedge \omega_n)_p\,.$$

Now, as $y_j = \langle Y, X_j \rangle$, we get $d_p y_j = \langle \nabla Y|_{T_pM}, X_j \rangle$ and hence

$$dy_j = \sum (X_i y_1) \omega_i \,.$$

Thus we get

$$d(i_Y d\mathrm{vol}_M)_p = (\sum \langle \nabla_{X_i} Y, X_i \rangle)_p$$

and conclude $\operatorname{div} Y = \operatorname{tr} \nabla Y$. □

In $\mathbb{R}^n$, the divergence just defined is consistent with the usual definition: $\operatorname{div} Y = \sum_i \partial_i y_i$.

By Stokes' theorem we immediately obtain the following theorem.

Theorem 11.58 (Divergence theorem). *Let M be a compact oriented Riemannian manifold with boundary and $Y \in \Gamma TM$. Then*

$$\int_M \operatorname{div} Y = \int_{\partial M} \langle N, Y \rangle \,,$$

where N is the outward-pointing unit normal along the boundary.

Note: $\operatorname{div} Y$ is independent of (local) orientation and thus makes also makes sense for non-orientable Riemannian manifolds.

Definition 11.59. *Let M be a Riemannian manifold. Then the Laplacian $\Delta \colon C^\infty M \to C^\infty M$ of M is given by*

$$\Delta f = \operatorname{div} \operatorname{grad} f \,.$$

Clearly, $\Delta f = \operatorname{tr} \operatorname{Hess} f$.
Let M be a compact oriented manifold, then we can define a scalar product on $C^\infty M$ as follows

$$\langle\langle f, g \rangle\rangle = \int_M f \cdot g \,.$$

Theorem 11.60. *For all $f, g \in C^\infty M$,*

$$\langle\langle \Delta f, g \rangle\rangle = - \int_M \langle \operatorname{grad} f, \operatorname{grad} g \rangle = \langle\langle f, \Delta g \rangle\rangle \,.$$

This follows immediately the divergence theorem together with the following useful lemma.

Lemma 11.61. *For $f \in C^\infty M$ and $Y \in \Gamma TM$,*

$$\operatorname{div}(fY) = \langle \operatorname{grad} f, Y \rangle + f \operatorname{div} Y \,.$$

Proof. This follows from the definition of div and the fact that

$$X^\flat \wedge i_X d\mathrm{vol}_M = \langle X, Y \rangle d\mathrm{vol}_M$$

which is easy to verify. □

Theorem 11.62. *Let M be a compact oriented Riemannian manifold. Then*

$$\Delta f = 0 \Leftrightarrow f = const.$$

Proof. That's an immediate consequence of the last theorem. □

Let us look for a moment at the finite-dimensional case.

Theorem 11.63. *Let V be a finite-dimensional Euclidean vector space,* $\dim V = n$*, and* $A\colon V \to V$ *be linear. Then*

$$\operatorname{im} A = (\ker A^*)^{\perp}.$$

Proof. Let $x \in \operatorname{im} A$, $w = Av$, and $u \in \ker A^*$. Then

$$\langle w, u \rangle = \langle Av, u \rangle = \langle v, A^* u \rangle = 0.$$

Hence we have $\operatorname{im} A \subset (\ker A^*)^{\perp}$. To see equality, we choose an orthonormal basis. Then A is represented by a matrix and A^* is its transpose. Thus we find that $\dim \operatorname{im} A = \dim \operatorname{im} A^*$ and the dimension theorem yields

$$\dim \ker A^* = n - \dim \operatorname{im} A^* = n - \dim \operatorname{im} A = \dim (\operatorname{im} A)^{\perp}.$$

Thus we have equality. □

It is hard work to show that the previous theorem holds also for the case that $V = C^\infty M$ and $A = \Delta$ (seee e.g. Warner's 'Foundations of differentiable manifolds and Lie groups'). We just take this here for granted:

Theorem 11.64. *Let M be a connected oriented compact Riemannian manifold and* $g \in C^\infty M$*. Then:*

$$\exists f \in C^\infty M\colon \Delta f = g \Leftrightarrow \int_M g = 0.$$

Proof. $g \in \operatorname{im} \Delta \Leftrightarrow g \perp \ker \Delta = \mathbb{R}\mathbf{1} \Leftrightarrow 0 = \langle\langle g, \mathbf{1} \rangle\rangle = \int_M g \cdot \mathbf{1} = \int_M g.$ □

In fact one can prove (see Warner as well) that

a) There is a complete orhtonormal system $f_1, f_2, \ldots \in C^\infty M$ of eigenfunctions.

b) All eigenspaces $E_\lambda = \{f \in C^\infty M \mid \Delta f = \lambda f\}$ are finite-dimensional.

E.g. on the 2-sphere $M = \mathbb{S}^2$, the smallest eigenvalue is (as always) zero and the corresponding eigenspace consists of constants. The next eigenspace consists of restrictions of linear functions and the eigenspaces for larger eigenvalues consist of so called spherical harmonics.
On the circle $M = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ the Laplacian is given by the second derivative $\Delta f = f''$. Hence the eigenfunctions with respect to an eigenvalue λ must be of the form $\cos\left(\sqrt{n}t\right)$ and $\sin\left(\sqrt{n}t\right)$. So for $\sqrt{n} \in \mathbb{Z}$. Thus the eigfenvalues are $\lambda = n^2$ with $n \in \mathbb{Z}$. A complete orthonormal system of eigenfunctions is then given by the Fourier basis

$$\tfrac{1}{2\pi}\cos(nt), \quad \tfrac{1}{2\pi}\sin(nt), \quad n \in \mathbb{Z}.$$

This can be easily used to solve the heat equation $\dot{f} = \Delta f$ or the wave equation $\ddot{f} = \Delta f$ on the circle.

Applications to surfaces: Let M be a Riemannian surface with metric $\langle .,. \rangle$ and $\tilde{g} = e^{2u}\langle .,. \rangle$ be a conformally equivalent metric. We leave it as an exercise to show that the Gauß-curvatures satisfy the equation

$$\Delta u = K - e^{2u}\tilde{K}\,.$$

Conversely, on a torus we have $\int_M K = 0$ and we can alsways solve for u such that $\tilde{K} = 0$. Thus we have the following theorem.

Theorem 11.65. *Each Riemannian 2-dimensional torus is conformally flat.*

11.5.1 Hodge-star operator

Let V be an oriented Euclidean vector space, $\dim V = n < \infty$. Then we have a unique volume form $\det \in \Lambda^n V^*$ such that for any positively oriented orthonormal basis $X_1, \dots, X_n$

$$\det(X_1, \dots, X_n) = 1\,.$$

Choose an orthonormal basis $X_1, \dots, X_n \in V$ and let $\omega_1, \dots, \omega_n \in V^*$ denote its dual basis, i.e. $\omega_j = X_j^\flat$. Then

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}, \quad 1 \le i_1 < \cdots i_k \le n$$

forms a basis of $\Lambda^k V^*$. Define a Euclidean inner product $\langle .,. \rangle$ on $\Lambda^k V^*$ by demanding this basis to be orthonormal.

Theorem 11.66. *This Euclidean inner product on $\Lambda^k V^*$ is independent of the choice of $X_1, \dots, X_n$.*

Proof. Just a few pages... □

Theorem 11.67. *Let $\eta \in \Lambda^k V^*$. Then there is a unique $*\eta \in \Lambda^{n-k} V^*$ such that*

$$\omega \wedge *\eta = \langle \omega, \eta \rangle \det$$

for all $\omega \in \Lambda^k V^*$.

Proof. Consider the map $\varphi \Lambda^k V^* \to (\Lambda^{n-k} V^*)^*$ defined by $\omega \wedge \alpha = \varphi(\alpha)\det)$. This map is an isomorphism as one can check on a basis. Now define $*\eta = \varphi^{-1}(\omega \mapsto \langle \omega, \alpha \rangle)$. □

Example 11.68. E.g., if $\dim V = 5$. Then $*(\omega_1 \wedge \omega_2) = -\omega_3 \wedge \omega_4 \wedge \omega_5$.

For $\omega \in \Lambda^k V^*$ a basis form, then $\langle \omega \wedge *\omega = \langle \omega, \omega \rangle \det = \omega_1 \wedge \cdots \wedge \omega_n$.

Theorem 11.69. *Let $\omega \in \Lambda^k V^*$. Then $** \omega = (-1)^{k(n-k)}\omega$.*

Proof. Check this on a basis. E.g. if $\omega = \omega_1 \wedge \omega_k$, then $*\omega = \omega_{k_1} \wedge \cdots \wedge \omega_n$. Thus

$$\det = \omega \wedge *\omega = (\omega_{k+1} \wedge \cdots \wedge \omega_n) \wedge (-1)^{k(n-k)}(\omega_1 \wedge \omega_k) = *\omega \wedge ** \omega\,.$$

□

This means:

$$** \, \omega = \omega \quad \textit{unless} \begin{cases} n \textit{ is even} \\ k \textit{ is odd} \end{cases}$$

Example 11.70. Let M is an oriented Riemannian surface and J denote the 90-degree rotation in the posotive sense, $J^2 = -1$, $J^* = -J$, $\langle JX, Y \rangle = d\mathrm{vol}_M$. Let $X \in T_pM$, $|X| = 1$. Then X, JX is a positively oriented orthonormal basis of T_pM. Claim: For $\omega \in \Omega^1 M$ we have

$$*\omega = -\omega \circ J\,.$$

Example 11.71. Let $M = \mathbb{R}^3$. Then $*f = f \det$, $*dx = dy \wedge dz$, $*dy = dz \wedge dx$, $*dz = dx \wedge dy$ and $** = \mathrm{Id}$.

Theorem 11.72. *Let $\omega \in \Omega^1 M$. Then $*\omega = i_{\omega^\sharp} d\mathrm{vol}_M$.*

Proof. We can assume that $\omega = \omega_1$, where ω_j is a positively oriented orthonormal basis. Then one easily verifies that

$$*\omega = \omega_2 \wedge \omega_n = i_{\omega_1^\sharp}(\omega_1 \wedge \cdots \wedge \omega_n)$$

— check on oriented $(n-1)$-tuples. □

11.5.2 Combining $*$ and d

Definition 11.73. *Define $\delta \colon \Omega^k M \to \Omega^{k-1} M$ by*

$$\delta\omega := (-1)^{k+k(n-k)} * d * \omega = (-1)^k *^{-1} d * \omega\,.$$

Theorem 11.74. *Let M be compact. Then $\delta = d^*$, i.e. for all $\omega \in \Omega^k M, \eta \in \Omega^{k+1} M$ we have*

$$\int_M \langle d\omega, \eta \rangle = \int_M \langle \omega, \delta\eta \rangle\,.$$

Proof. This follows by Stokes' theorem:

$$\int_M \langle \omega, \delta\eta \rangle = \int_M \omega \wedge *\delta\eta = \int_M (-1)^{k+1} \omega \wedge d * \eta = \int_M d\omega \wedge *\eta = \int_M \langle d\omega, \eta \rangle\,.$$

□

Theorem 11.75. *Let M be a compact oriented Riemannian manifold. Consider the restrictions*

$$d_k \colon \Omega^k M \to \Omega^{k+1} M, \quad \delta_k \colon \Omega^{k-1} M \to \Omega^k M\,.$$

Then

$$\mathrm{im}\, d_k = (\ker \delta_k)^\perp \quad \mathrm{im}\, \delta_k = (\ker d_{k-1})^\perp$$

Proof. See Warner. □

The space of *harmonic* forms is defined as the intesection of closed and co-closed forms

$$\mathrm{harm}_k M := \ker d_k \cap \ker \delta_k \,.$$

Theorem 11.76. $\dim \mathrm{harm}_k M < \infty$ *and*

$$\Omega^k M = \mathrm{im}\, d_{k-1} \oplus_\perp \mathrm{harm}_k\, M \oplus_\perp \mathrm{im}\, \delta_{k+1} M \,.$$

Proof. Easy to check. Hint: $\mathrm{im}\, d_{k_1} \oplus \mathrm{harm}_k\, M = \ker d_k$, $\mathrm{harm}_k\, M \oplus \mathrm{im}\, \delta_{k+1} M = \ker \delta_k$. ☐